A counterexample to Oertel's conjecture on mixed-integer centerpoints

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Abstract

Grünbaum's inequality states that every closed halfspace containing the centroid of a convex body in $\mathbb{R}^d$ contains at least a $(d/(d+1))^d$ fraction of its volume. Oertel conjectured a mixed-integer analogue: every convex body $C \subset \mathbb{R}^n \times \mathbb{R}^d$ of positive mixed-integer volume (the sum of the d-dimensional volumes of its fibers over $\mathbb{Z}^n$) contains a point of $\mathbb{Z}^n \times \mathbb{R}^d$ such that every closed halfspace containing it carries at least a $2^{-n}(d/(d+1))^d$ fraction of this volume. The product of the cube $[0,1]^n$ with a d-simplex shows that this would be best possible. We disprove the conjecture for $(n,d) = (12,5)$ by constructing a rational polytope in which every point of $\mathbb{Z}^{12} \times \mathbb{R}^5$ lies in a closed halfspace carrying strictly less than this fraction. The construction circumvents the parity obstruction that underlies the conjectured bound: the fiber over the origin is the midpoint fiber of 756 antipodal pairs of fibers, and three halfspaces covering it each pick up only a third of the small fibers that pay for this. The gap to the conjectured bound is small (a relative $6.5 \cdot 10^{-9}$), and all comparisons are carried out in exact rational arithmetic. The existence of a convex body with this property is also verified in Lean 4.

1 Introduction

Cutting plane methods for convex minimization measure progress by the volume that a cut removes from the search region. A query point with a fixed fraction of the volume in every closed halfspace containing it therefore guarantees progress whatever the direction of the cut, and the following notions measure this fraction when some of the variables are integer.

Mixed-integer volume and depth. We write points of $\mathbb{R}^n \times \mathbb{R}^d$ as (z,x). For a set $D \subseteq \mathbb{R}^n \times \mathbb{R}^d$ and $z \in \mathbb{Z}^n$, the *fiber* of D over z is $D_z = \{x \in \mathbb{R}^d : (z,x) \in D\}$, and the *mixed-integer volume* of D is

$$\mu(D) = \sum_{z \in \mathbb{Z}^n} \mathrm{vol}_d(D_z),$$

where vol_d is Lebesgue measure on $\mathbb{R}^d$ (and counting measure if $d = 0$). By a *halfspace* we always mean a closed halfspace $\{(z,x) : F(z,x) \geq b\}$ with F a nonzero linear functional on $\mathbb{R}^n \times \mathbb{R}^d$ and $b \in \mathbb{R}$. For a convex body C with $\mu(C) > 0$, the *depth* of a point $p \in \mathbb{Z}^n \times \mathbb{R}^d$ is

$$\mathrm{depth}_C(p) = \inf\Big\{ \tfrac{\mu(C \cap H)}{\mu(C)} : H \text{ a halfspace with } p \in H \Big\}.$$

Points outside C have depth 0, by strict separation. Since C is compact, it has finitely many nonempty fibers, so $\mu(C)$ is finite, and restricting H to halfspaces with p on their boundary, as in [1, (2)], does not change the infimum. We call $\mu(C)$ the *mass* of C and $\mu(C \cap H)$ the mass of H (Fig. 1(a)).

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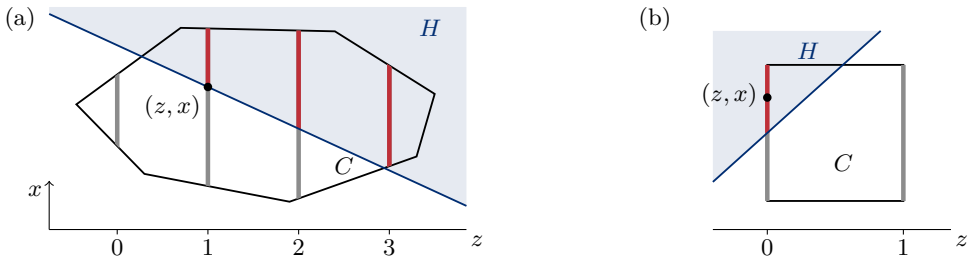


Figure 1: Depth for $n = d = 1$. (a) The fibers of C are its vertical chords over the integers, and $\mu(C)$ is their total length. The halfspace H contains the point (z, x), and the red segments are the fibers of $C \cap H$. (b) In the product $C = [0,1] \times [0,1]$, the halfspace through $(0, 1/2)$ is tilted to miss the fiber over 1. It contains every point $(0, x)$ with $x \geq 1/2$ and carries one quarter of the mass.

For $n = 0$, μ is ordinary volume, and Grünbaum's theorem [5, Theorem 2 and Section 3] states that the centroid of a convex body in $\mathbb{R}^d$ has depth at least

$$\Gamma_d = \left(\frac{d}{d+1}\right)^d,$$

with equality for simplices [5, Section 4]. For $d = 0$, μ counts integer points, and the depth of the best integer point can be as small as 2^{-n} (take $C = [0,1]^n$). Basu and Oertel [1, Section 2 and Theorem 2.6] showed how lower bounds on the depth of a point of $\mathbb{Z}^n \times \mathbb{R}^d$ yield bounds on the number of oracle calls of cutting plane methods for convex mixed-integer optimization, and proved the general lower bound $2^{-n}/(d+1)$; see also [7, Theorem 4.1.19] and [1, Corollary 3.4].

The two extreme cases combine in the product $C = [0,1]^n \times S$ of the unit cube and a d-simplex S [7, Section 4.1, after Conjecture 4.1.20]. Here every fiber over $\{0,1\}^n$ is a copy of S and carries a 2^{-n} fraction of the mass. A *Grünbaum cap* of S is the set where one of its barycentric coordinates is at least $1/(d+1)$, a homothetic copy of S with ratio $d/(d+1)$ and relative volume Γ_d, and every point of S lies in such a cap. A point (z, x) with $z \in \{0,1\}^n$, $x \in S$ lies in a halfspace that meets the fiber over z in a Grünbaum cap of S and misses all other fibers; to see this, add to the cap's defining functional a large multiple of a linear functional on $\mathbb{R}^n$ that is uniquely maximized over $\{0,1\}^n$ at z (Fig. 1(b)). Conversely, the fiber over z of every halfspace containing $(z, \text{centroid of } S)$ contains a halfspace of $\mathbb{R}^d$ through this centroid. Hence the largest depth in this example is exactly $2^{-n}\Gamma_d$. Oertel conjectured that this example is extremal.

Conjecture 1.1 (Oertel [7, Conjecture 4.1.20], Basu–Oertel [1, Conjecture 3.5])**.** *For every convex body $C \subset \mathbb{R}^n \times \mathbb{R}^d$ with $\mu(C) > 0$ there is a point $p \in \mathbb{Z}^n \times \mathbb{R}^d$ with* $\text{depth}_C(p) \geq 2^{-n}\Gamma_d$ *(where $\Gamma_0 = 1$).*

The conjecture has been confirmed under additional assumptions that make the projection of C onto $\mathbb{R}^n$ large: Basu and Oertel [1, Theorem 3.6] proved it when this projection has lattice width above a threshold exponential in n, and Cristi and Salas [4, Theorem 4.5] when it contains a Euclidean ball of radius at least $\alpha d^2 n^{3/2}$ for a universal constant α. Cheng and Basu [2, Corollary 2.5] showed depth at least $1/(2e)$ when the projection contains a translate of an ℓ_∞-ball of radius $k \geq 3e(n+d)$, and [3, Theorem 1] proved the conjectured value $2/9 = \Gamma_2/2$ for compact convex sets in $\mathbb{R} \times \mathbb{R}^2$ with exactly three fibers of positive area. Our result shows that some such assumption is needed. The projection onto $\mathbb{R}^{12}$ of the polytope in Theorem 1.2 lies in $[-1,1]^{12}$.

Theorem 1.2. *There is a full-dimensional rational polytope $C \subset \mathbb{R}^{12} \times \mathbb{R}^5$ and a rational number*

$$r < 2^{-12}\Gamma_5 = 2^{-12}(5/6)^5$$

such that every point of $\mathbb{Z}^{12} \times \mathbb{R}^5$ *lies in a halfspace* H *with* $\mu(C \cap H) \le r\,\mu(C)$. *In particular,* $\operatorname{depth}_C(p) \le r$ *for all* $p \in \mathbb{Z}^{12} \times \mathbb{R}^5$, *and Conjecture 1.1 fails for* $(n, d) = (12, 5)$.

Numerically, $10^5 r = 9.81146409\ldots$, whereas $10^5 \cdot 2^{-12}\Gamma_5 = 9.81146416\ldots$; the relative gap is about $6.5 \cdot 10^{-9}$. All strict inequalities in the proof are comparisons of explicit rational numbers, which we verified in exact arithmetic (Section 6); the bound on the depth has also been formalized in Lean 4 (Appendix A).

Organization. Section 2 explains the idea of the construction and how the mass is accounted for; it can be read on its own. Section 3 defines the integer points carrying fibers, Section 4 the fibers themselves, Section 5 constructs the halfspaces, and Section 6 fixes the parameters and completes the proof.

2 Overview of the construction

Throughout, $n = 12$, $d = 5$, and $\Gamma = \Gamma_5 = (5/6)^5$. Our polytope will be $C = \operatorname{conv} \bigcup_{z \in \mathcal{Z}} \{z\} \times A_z$ for a finite set $\mathcal{Z} \subset \mathbb{Z}^{12}$ and polytopes $A_z \subset \mathbb{R}^5$, arranged so that the fiber of C over z is exactly A_z for $z \in \mathcal{Z}$, and empty for all other integer z.

Two basic observations. First, the argument for the cube shows that if z is a vertex of $\operatorname{conv}(\mathcal{Z})$, then any halfspace G of $\mathbb{R}^5$ can be *tilted* into a halfspace of $\mathbb{R}^{12} \times \mathbb{R}^5$ that meets the fiber over z in $A_z \cap G$ and misses all other fibers. If A_z is a simplex, then every point of $\{z\} \times A_z$ lies in a halfspace of mass $\Gamma \operatorname{vol}(A_z)$, by tilting a Grünbaum cap of A_z.

Second, the argument for the converse shows that the centroid of any fiber A_z has depth at least $\Gamma \operatorname{vol}(A_z)/\mu(C)$. So in a counterexample every fiber has volume less than $2^{-n}\mu(C)$, and hence more than 2^n fibers have positive volume. Two of them then lie over points in the same class of $\mathbb{Z}^n/2\mathbb{Z}^n$, so their midpoint is an integer point whose fiber contains the average of the two. This is the obstruction behind the conjecture: not every fiber can be isolated by tilting. The polytope of Theorem 1.2 has $4852 > 2^{12}$ fibers, and we must deal with the fibers that cannot be isolated.

The budget. Fix a parameter $h > 0$ and put $M = \Gamma h^5$, the mass of a tilted Grünbaum cap of a simplex fiber of volume h^5. We will show that every point of $\mathbb{Z}^{12} \times \mathbb{R}^5$ lies in a halfspace of mass at most M. Then every depth is at most $M/\mu(C)$, and Theorem 1.2 follows from

$$\mu(C) > 2^{12} h^5 = 4096\, h^5. \tag{1}$$

In words: on average, each of the 4096 classes of $\mathbb{Z}^{12}/2\mathbb{Z}^{12}$ must contribute more mass than a single isolated simplex fiber whose caps fit the budget.

Heavy fibers. In each of the 4095 nonzero classes we place one *heavy* point $z \in \mathcal{Z}$, which is a vertex of $\operatorname{conv}(\mathcal{Z})$. Its fiber has volume at most h^5 and is covered by tilted caps of mass at most M. The class of 0 contains the origin, whose fiber is a 5-simplex B; we normalize volumes so that $\operatorname{vol}(B) = 1$. Our parameters have $h^5 \approx 1.69988 > 1$, so these fibers alone do not satisfy (1): the class of the origin must make up for a shortfall of $h^5 - 1$.

The origin cluster. The additional mass comes from $3 \cdot 252 = 756$ small *light* fibers, located at the points $-z$ for 756 of the heavy points z (we call these heavy points *paired*). Since 0 is the midpoint of z and $-z$, the origin is not a vertex of $\operatorname{conv}(\mathcal{Z})$ and its fiber cannot be isolated. Instead we cover it, together with all light fibers, by three halfspaces H_1, H_2, H_3 with the following properties (Fig. 2(a)).

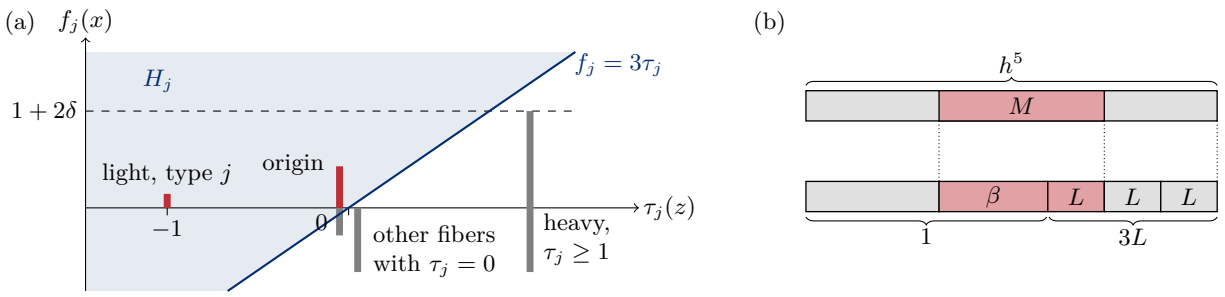


Figure 2: (a) Schematic of the halfspace H_j over the origin cluster, in the plane with coordinates $(\tau_j(z), f_j(x))$. Each vertical bar is the combined range of f_j on a group of fibers, and the parts inside H_j are dark red. H_j contains the light fibers of type j and the cap $B \cap \{f_j \geq 0\}$ of the origin fiber. It meets the other fibers over $\{\tau_j = 0\}$ only in the hyperplane $\{f_j = 0\}$ and misses all fibers over $\{\tau_j \geq 1\}$, since $|f_j| \leq 1 + 2\delta < 3$ on heavy fibers. (b) Volumes to scale. Top: a heavy fiber of volume h^5 and a tilted Grünbaum cap of mass M. Bottom: the origin fiber, of volume 1, and the three pools, of volume L each. Red marks what the cap and H_1 carry.

(i) The traces of H_1, H_2, H_3 on the origin fiber cover B, and each has volume $\beta \approx 0.44976$.

(ii) The light fibers are split into three pools of 252, of total volume $L \approx 0.23339$ each, and H_j contains the j-th pool entirely.

(iii) H_j meets every other fiber in a set of volume zero.

Each H_j then has mass $\beta + L$, and we need $\beta + L \leq M$. The cluster consisting of the origin fiber and the three pools contributes $1 + 3L$ to $\mu(C)$, while each of its points is covered at cost at most $\beta + L$ (Fig. 2(b)). Compared with an isolated simplex fiber, whose mass is $\Gamma^{-1} = (6/5)^5 \approx 2.48832$ times the cost of its caps, the cluster does slightly better:

$$\frac{1 + 3L}{\beta + L} \approx 2.48874 > (6/5)^5.$$

This is the only source of gain in the construction. It explains the number three: two halfspaces covering B have traces of total volume at least 1, so with two pools the corresponding ratio $(1 + L_1 + L_2)/\max_j(\beta_j + L_j)$ is at most $2 < \Gamma^{-1}$, whereas with three pools it can approach 3.

The price. Two requirements tie the light fibers to their heavy partners and make the paired fibers slightly smaller than h^5.

- *The origin fiber must stay B.* The fiber of C over 0 contains $\frac{1}{2}(A_z + A_{-z})$ for each paired z. We ensure that it is exactly B by requiring $A_z \subseteq (1 + g(z))B$ for a linear function g (dilations are about the centroid of B, placed at the origin). Paired fibers then lie in $(1 + \delta)B$ and light fibers in $(1 - \delta)B$, where $\delta = g(z)$ for paired z.

- *H_j is a single halfspace.* We take $H_j = \{(z, x) : f_j(x) \geq 3\tau_j(z)\}$, where f_j is a linear functional on $\mathbb{R}^5$ with $\{f_j \geq 0\} \cap B$ the cap in (i), and τ_j is an integer-valued linear form on $\mathbb{Z}^{12}$. Over a point $z \in \mathcal{Z}$, the trace of H_j is $\{f_j \geq 3\tau_j(z)\}$. Requirements (ii) and (iii) are therefore met if $\tau_j = -1$ on the j-th pool, $\tau_j \geq 1$ wherever possible, and $A_z \subseteq \{f_j \leq 0\}$ wherever $\tau_j(z) = 0$. Since τ_j is linear, $\tau_j(z) = -\tau_j(-z)$. So each paired fiber is forced into the same region as its light partner, namely a dilate of $T_j = B \cap \{f_i \leq 0 \text{ for } i \neq j\}$. Linearity also forces $\tau_j(z) = 0$

for some other heavy points (we call them *one-sided*); their fibers must avoid one cap of B, which costs nothing because they are simplices of volume h^5 anyway.

A paired fiber must lie in $(1+\delta)T_j$ and be coverable by halfspaces of mass at most M. The natural candidate is a dilated simplex $(1+\delta)U_j$ with $U_j \subset T_j$, but its volume ≈ 1.69640 falls short of $h^5 \approx 1.69988$. Over 756 paired points this deficit would far exceed the gain $3L \approx 0.700$. We therefore add a thin slab to U_j, which reduces the deficit to about $3 \cdot 10^{-7}$ per paired fiber while keeping the covering caps within budget. The final account is

$$\mu(C) - 4096\, h^5 = \underbrace{(1 + 3L - h^5)}_{\approx 2.82\cdot 10^{-4}} - \underbrace{756\big(h^5 - \mathrm{vol}(\text{paired fiber})\big)}_{\approx 2.36\cdot 10^{-4}} > 0, \tag{2}$$

where all other heavy fibers have volume exactly h^5.

Why so many integer dimensions. The pools must be large: their total volume $3L$ has to exceed $h^5 - 1 \approx 0.7$, while each light fiber is tiny (volume $(1-\delta)^5 \,\mathrm{vol}(T_j) \approx 9.3\cdot 10^{-4}$, since $\delta \approx 0.664$ is large in order to make the paired fibers big). We obtain $252 = \binom{10}{5}$ paired points of each type from a 10-dimensional Boolean cube, with two extra integer coordinates distinguishing the three types. Each heavy point lies in its own class modulo 2, so the number of heavy points, and hence of paired points, is limited by $2^{12} - 1$.

3 The integer points

We write $z = (y, s) \in \mathbb{R}^{10} \times \mathbb{R}^2$, put $u(z) = y_1 + \cdots + y_{10}$, and define the linear maps

$$\tau(z) = (\tau_1, \tau_2, \tau_3)(z) = (u - s_2,\ u - s_1,\ u - s_1 - s_2), \qquad \sigma(z) = \tau_1 + \tau_2 + \tau_3 = 3u - 2s_1 - 2s_2.$$

Let e_1, e_2 be the unit vectors of $\mathbb{R}^2$ and $\mathbf{1} = (1,1,1)$; inequalities between vectors are meant componentwise, and $[k] = \{1, \ldots, k\}$.

Representatives of the classes modulo 2. For $J \subseteq [10]$ let $\mathbf{1}_J \in \{0,1\}^{10}$ be its indicator vector and let $b_J \in \{-1, 0, 1\}^{10}$ be the signed indicator vector with entries

$$(b_J)_i = \begin{cases} -1, & i \in J,\ i \le 4, \\ 1, & i \in J,\ i \ge 5, \\ 0, & i \notin J. \end{cases}$$

Then $b_J \equiv \mathbf{1}_J \pmod 2$ and $u(b_J) = |J \setminus [4]| - |J \cap [4]| = |J \triangle [4]| - 4$. Since $J \mapsto J \triangle [4]$ is a bijection,

$$\#\{J \subseteq [10] : u(b_J) = k\} = \binom{10}{k+4}. \tag{3}$$

The signs serve to center this distribution at $u = 1$, the value that will produce paired points; the unsigned vectors would give only 10 sets with $u(\mathbf{1}_J) = 1$ instead of $\binom{10}{5} = 252$.

The 4095 points

$$(\mathbf{1}_J, 0) \quad (\emptyset \ne J \subseteq [10]) \qquad \text{and} \qquad (b_J, s) \quad \big(J \subseteq [10],\ s \in \{e_1, e_2, e_1 + e_2\}\big)$$

represent the 4095 nonzero classes of $\mathbb{Z}^{12}/2\mathbb{Z}^{12}$, each exactly once, because $e_1, e_2, e_1 + e_2$ are the three nonzero classes of $\mathbb{Z}^2/2\mathbb{Z}^2$. On each of them $\sigma \ne 0$: we have $\sigma(\mathbf{1}_J, 0) = 3|J| > 0$, and for the others $s_1 + s_2 \in \{1, 2\}$, so $\sigma \equiv -2(s_1 + s_2) \not\equiv 0 \pmod 3$. The *heavy points* are these representatives multiplied by the sign of σ, so that $\sigma > 0$ on every heavy point. Negation preserves the class modulo 2, so the heavy points still represent each nonzero class exactly once.

point $z \in \mathcal{Z}$	number	$\tau(z)$	$\sigma(z)$	fiber A_z (Section 4)
origin	1	0	0	B
paired, type j	3×252	e_j	1	$(1+\delta)K_j$
light, type j	3×252	$-e_j$	-1	$(1-\delta)T_j$
one-sided, type j	3×210	$\mathbf{1} - e_j$	2	$(1+2\delta)N_j$
free	2709	$\geq \mathbf{1}$	≥ 3	hB

Table 1: The 4852 points of $\mathcal{Z}$.

Lemma 3.1. *Every heavy point z satisfies $\tau(z) \geq 0$, and exactly one of the following holds:*

- *$\tau(z) = e_j$ for some $j \in [3]$ (here $e_j \in \mathbb{R}^3$): z is* paired *of type j;*
- *$\tau(z) = \mathbf{1} - e_j$ for some $j \in [3]$: z is* one-sided *of type j;*
- *$\tau(z) \geq \mathbf{1}$: z is* free.

For each j there are 252 paired and 210 one-sided points of type j, and there are 2709 free points.

Proof. For $(\mathbf{1}_J, 0)$ we have $\tau = |J|\mathbf{1} \geq \mathbf{1}$. For $s \in \{e_1, e_2, e_1+e_2\}$ and $u = u(b_J)$, the following table lists τ at the heavy point $\pm(b_J, s)$ (the sign being that of $\sigma(b_J, s)$).

s	$\tau(b_J, s)$	heavy point	τ at heavy point	paired	one-sided
e_1	$(u, u-1, u-1)$	$\begin{cases}(b_J,s), & u \geq 1\\ -(b_J,s), & u \leq 0\end{cases}$	$\begin{cases}(u, u-1, u-1)\\ (-u, 1-u, 1-u)\end{cases}$	$u=1$	$u=0$
e_2	$(u-1, u, u-1)$	$\begin{cases}(b_J,s), & u \geq 1\\ -(b_J,s), & u \leq 0\end{cases}$	$\begin{cases}(u-1, u, u-1)\\ (1-u, -u, 1-u)\end{cases}$	$u=1$	$u=0$
e_1+e_2	$(u-1, u-1, u-2)$	$\begin{cases}(b_J,s), & u \geq 2\\ -(b_J,s), & u \leq 1\end{cases}$	$\begin{cases}(u-1, u-1, u-2)\\ (1-u, 1-u, 2-u)\end{cases}$	$u=1$	$u=2$

In every row $\tau \geq 0$ at the heavy point; it equals e_j (with $j = 1, 2, 3$ in the three rows) for the value of u in the column "paired", it equals $\mathbf{1} - e_j$ for the value in the column "one-sided", and it is $\geq \mathbf{1}$ otherwise. By (3) the paired points of each type number $\binom{10}{5} = 252$, and the one-sided points of each type number $\binom{10}{4} = \binom{10}{6} = 210$. The remaining $1023 + 3 \cdot (1024 - 252 - 210) = 2709$ heavy points are free. □

For example, for $J = \{1, 5, 6\}$ we have $u(b_J) = 1$, so (b_J, e_1) and (b_J, e_2) are paired of types 1 and 2, while $-(b_J, e_1 + e_2)$ is paired of type 3, and $(\mathbf{1}_J, 0)$ is free (Fig. 3).

Light points and the set $\mathcal{Z}$. For each paired point z of type j, the point $-z$ is a *light point* of type j; it satisfies $\tau(-z) = -e_j$. Let $\mathcal{Z}$ consist of 0, the 4095 heavy points and the 756 light points. These are 4852 distinct points: a light point $-z$ lies in the class of z modulo 2, but differs from z, the only heavy point in that class. Table 1 summarizes the values of τ and σ. In particular, for $z \in \mathcal{Z}$ and $j \in [3]$,

$$\tau_j(z) < 0 \iff z \text{ is light of type } j, \text{ and then } \tau_j(z) = -1. \tag{4}$$

The following lemma shows that $\mathcal{Z}$ creates no further integer points in its convex hull, and that every nonzero point of $\mathcal{Z}$ can be isolated.

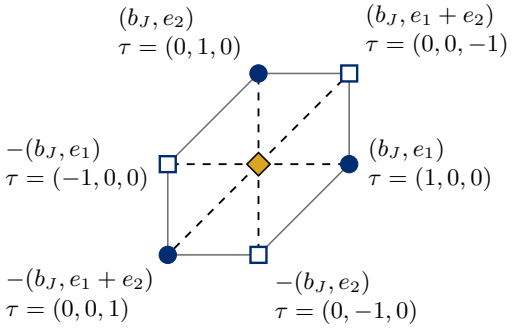


Figure 3: The points of $\mathcal{Z}$ obtained from $J = \{1,5,6\}$ and the origin, plotted by their coordinates s. Circles are paired points, squares are light points, and the diamond marks $s = 0$, shared by the origin, with $\tau = 0$, and the free point $(\mathbf{1}_J, 0)$, with $\tau = 3 \cdot \mathbf{1}$. Dashed segments join antipodal points. The hexagon is the convex hull of the coordinates s of all points of $\mathcal{Z}$.

Lemma 3.2. *If $z \in \mathbb{Z}^{12} \setminus \{0\}$ is a convex combination $z = \sum_m \lambda_m z_m$ of points $z_m \in \mathcal{Z}$ with all $\lambda_m > 0$, then $z_m = z$ for all m. Consequently* $\mathrm{conv}(\mathcal{Z}) \cap \mathbb{Z}^{12} = \mathcal{Z}$, *and every point of $\mathcal{Z} \setminus \{0\}$ is a vertex of* $\mathrm{conv}(\mathcal{Z})$.

Proof. We use two facts. First, all points of $\mathcal{Z}$ have entries in $\{-1,0,1\}$, and every point of $\mathcal{Z}$ with $s \neq 0$ has the form $\omega(b_J, s')$ with $\omega \in \{\pm 1\}$ and $s' \in \{e_1, e_2, e_1 + e_2\}$. Second, if an integer t is a convex combination with positive weights of integers t_m that all lie in $\{k, k+1\}$ for some k, then $t_m = t$ for all m. We apply the second fact coordinatewise. By the first fact, if a coordinate of z equals ± 1, then that coordinate of every z_m is the same.

Case $s \neq 0$. Say $s_k = \pm 1$. Then every z_m has the same k-th s-coordinate, so every z_m has the form $\omega(b_{J_m}, s'_m)$ with one common sign ω (the entries of s'_m are nonnegative). Hence the other s-coordinate of every z_m lies in $\{0, \omega\}$, and the i-th y-coordinate lies in $\{0, \omega(b_{[10]})_i\}$. By the second fact all z_m agree with z in every coordinate.

Case $s = 0$. Since $z \neq 0$, some $y_i = \pm 1$, and then every z_m has i-th y-coordinate $y_i \neq 0$. If $z_m = \omega_m(b_{J_m}, s'_m)$ has $s'_m \neq 0$, then $y_i = \omega_m(b_{[10]})_i$, so $\omega_m = \omega := y_i(b_{[10]})_i$ is the same for all such m. Thus $\omega(s_{m,1} + s_{m,2}) \geq 0$ for all m, with equality only if $s_m = 0$. Since $\sum_m \lambda_m(s_{m,1} + s_{m,2}) = s_1 + s_2 = 0$, all $s_m = 0$. The points of $\mathcal{Z}$ with $s = 0$ are 0 and the $(\mathbf{1}_J, 0)$, whose y-parts lie in $\{0,1\}^{10}$, so $y_m = y$ for all m by the second fact.

For the consequences, note that $0 \in \mathcal{Z}$, and that a nonzero point of $\mathcal{Z}$ in the convex hull of the other points of $\mathcal{Z}$ would have a representation with some $z_m \neq z$. □

Since every vertex of a polytope is exposed, for each $z \in \mathcal{Z} \setminus \{0\}$ we fix a linear functional ℓ_z on $\mathbb{R}^{12}$ with $\ell_z(z') < \ell_z(z)$ for all $z' \in \mathcal{Z} \setminus \{z\}$.

4 The fibers

4.1 The simplex and three cuts through its centroid

Let $B = \mathrm{conv}(v_1, \ldots, v_6) \subset \mathbb{R}^5$ with $v_i = e_i$ for $i \leq 5$ and $v_6 = -(e_1 + \cdots + e_5)$. This simplex has rational vertices and its centroid is the origin, so for $\lambda > 0$ the set λB is the dilation of B about its centroid. Let $x_1, \ldots, x_6$ be the barycentric coordinates of B, viewed as affine functions on $\mathbb{R}^5$ with $\sum_i x_i = 1$. *From now on,* vol *denotes Lebesgue measure on $\mathbb{R}^5$ normalized so that* $\mathrm{vol}(B) = 1$; this rescales μ by a constant and does not affect depths. We call $F = \mathrm{conv}(v_1, v_2, v_3)$ the *active face* and put $F' = \mathrm{conv}(v_4, v_5, v_6)$; let $\bar{v} = (v_1 + v_2 + v_3)/3$ be the centroid of F.

Fix a rational number a with $\frac{1}{2} < a < 1$. With indices $1, 2, 3$ taken cyclically, define

$$f_j = x_j - a\, x_{j+1} - (1-a)\, x_{j+2} \qquad (j = 1, 2, 3).$$

These functions have the following properties.

(a) f_j vanishes at the centroid ($x_i = 1/6$ for all i), so f_j is a linear functional on $\mathbb{R}^5$.

(b) $f_1 + f_2 + f_3 = 0$. Hence $B \cap \{f_1 \geq 0\}$, $B \cap \{f_2 \geq 0\}$ and $B \cap \{f_3 \geq 0\}$ cover B.

(c) f_j takes the values $1, -a, -(1-a)$ at v_j, v_{j+1}, v_{j+2} and 0 at v_4, v_5, v_6. In particular $|f_j| \leq 1$ on B.

(d) The linear map that permutes $v_1 \to v_2 \to v_3 \to v_1$ and fixes v_4, v_5, v_6 preserves B and volume, and maps the zero sets and sign regions of f_j to those of f_{j+1}.

By (d), it suffices to describe all sets below for $j = 1$; the sets for $j = 2, 3$ are their images under this symmetry.

All the f_j depend only on the *active coordinates* (x_1, x_2, x_3) and are homogeneous in them. The following two facts reduce volume computations to the triangle F.

Lemma 4.1. *(a) Let Q be the solution set in B of finitely many homogeneous linear inequalities in (x_1, x_2, x_3). Then $Q = \operatorname{conv}\big((Q \cap F) \cup F'\big)$.*

(b) If the active coordinates of $c_1, c_2, c_3 \in F$ form the rows of a 3×3 matrix P, then the convex hull of $c_1, c_2, c_3, v_4, v_5, v_6$ has volume $|\det P|$.

Proof. (a) A point $x \in Q$ with $\lambda = x_1 + x_2 + x_3 \in (0, 1)$ is the convex combination $\lambda x' + (1-\lambda)x''$ of the point $x' \in F$ with active coordinates $(x_1, x_2, x_3)/\lambda$ and a point $x'' \in F'$. By homogeneity $x' \in Q$. Points with $\lambda \in \{0, 1\}$ lie in F' or F. Conversely, $F' \subseteq Q$ because all active coordinates vanish there, and Q is convex. (b) The affine map sending $v_i \mapsto c_i$ ($i \leq 3$) and fixing v_4, v_5, v_6 acts on barycentric coordinates by the block matrix $\operatorname{diag}(P^\top, I_3)$, so it scales volume by $|\det P|$. □

The cap. On F, f_1 vanishes at $(av_1 + v_2)/(1+a)$ and at $((1-a)v_1 + v_3)/(2-a)$, so by Lemma 4.1 the cap $B \cap \{f_1 \geq 0\}$ is a simplex of volume

$$\beta = \det \begin{pmatrix} 1 & 0 & 0 \\ \frac{a}{1+a} & \frac{1}{1+a} & 0 \\ \frac{1-a}{2-a} & 0 & \frac{1}{2-a} \end{pmatrix} = \frac{1}{(1+a)(2-a)}.$$

The region T_1. Let $T_1 = B \cap \{f_2 \leq 0,\ f_3 \leq 0\}$. On T_1 we have $f_1 = -f_2 - f_3 \geq 0$. On F, the lines $\{f_2 = 0\}$ and $\{f_3 = 0\}$ pass through $\bar{v}$ and meet the edges v_1v_2 and v_1v_3 in

$$w_2 = \frac{v_1 + (1-a)v_2}{2-a}, \qquad w_3 = \frac{v_1 + av_3}{1+a},$$

respectively. So $T_1 \cap F$ is the quadrilateral $v_1 w_2 \bar{v} w_3$ (Fig. 4). Its diagonal $v_1\bar{v}$ lies on the line $\{x_2 = x_3\}$ and splits it into two triangles. By Lemma 4.1, $T_1 = U_1 \cup P_1$ with

$$U_1 = \operatorname{conv}(v_1, \bar{v}, w_3, v_4, v_5, v_6), \qquad P_1 = \operatorname{conv}(v_1, \bar{v}, w_2, v_4, v_5, v_6).$$

These simplices share the facet $\operatorname{conv}(v_1, \bar{v}, v_4, v_5, v_6)$ and lie on opposite sides of the hyperplane $\{x_2 = x_3\}$. Their volumes are

$$\theta = \operatorname{vol}(U_1) = \frac{a}{3(1+a)}, \qquad \varphi = \operatorname{vol}(P_1) = \frac{1-a}{3(2-a)}.$$

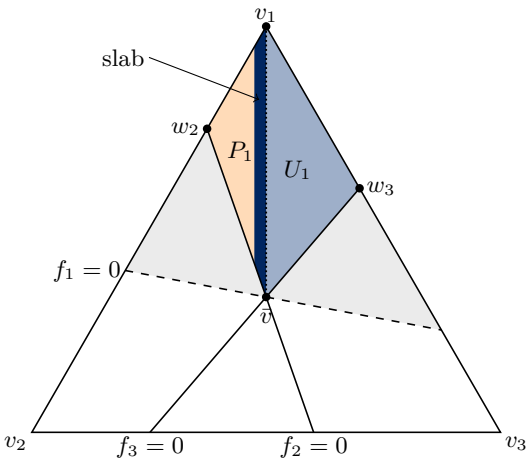


Figure 4: The active face $F = \mathrm{conv}(v_1, v_2, v_3)$, drawn for $a = 0.663$. The subsets of B in Section 4.1, except K_1 and its slab, are joins of their traces on F with $F' = \mathrm{conv}(v_4, v_5, v_6)$, so the picture shows traces only. The cap $B \cap \{f_1 \geq 0\}$ is the triangle above the dashed line. Inside it, T_1 is the quadrilateral $v_1 w_2 \bar{v} w_3$, split by the dotted diagonal into U_1 (blue) and P_1 (orange). The dark strip indicates the slab $P_1 \cap \{p \leq \varepsilon\}$ added to U_1 to form K_1; its width is exaggerated, since $\varepsilon = 6.5 \cdot 10^{-4}$. Note that the centroid of B does not lie on F.

The slab. The paired fiber will be a dilate of U_1 enlarged by a thin slab of P_1 along the common facet. Let $\nu_0, \dots, \nu_5$ be the barycentric coordinates of U_1 with respect to the vertex order $(v_1, \bar{v}, w_3, v_4, v_5, v_6)$, and let $q_i = \nu_i(w_2)$. Comparing active coordinates gives

$$q_1 = \frac{3(1-a)}{2-a}, \qquad q_2 = -\frac{1-a^2}{a(2-a)}, \qquad q_0 = 1 - q_1 - q_2, \qquad q_3 = q_4 = q_5 = 0.$$

Because $\frac{1}{2} < a < 1$, we have $0 < q_1 < 1$, $q_2 < 0$, and $q_1 + q_2 = \frac{(1-a)(2a-1)}{a(2-a)} \in (0, 1)$, hence $0 < q_0 < 1$. Let p be the barycentric coordinate of P_1 at w_2 and, for $i \neq 2$, let p_i be the barycentric coordinate of P_1 at the vertex that P_1 shares with U_1 in position i. Both sides of

$$\nu_i = p_i + q_i p \quad (i \neq 2), \qquad \nu_2 = q_2 p \tag{5}$$

are affine and agree at the vertices of P_1, so (5) holds on $\mathbb{R}^5$. In particular $\nu_2 \geq 0$ on U_1 and $\nu_2 \leq 0$ on P_1. Fix a rational $\varepsilon \in (0, 10^{-3})$ and define

$$K_1 = T_1 \cap \{\nu_2 \geq \varepsilon q_2\} = U_1 \cup \big(P_1 \cap \{p \leq \varepsilon\}\big).$$

This is a convex polytope. Since $P_1 \cap \{p \geq \varepsilon\}$ is a homothetic copy of P_1 with ratio $1 - \varepsilon$,

$$V := \mathrm{vol}(K_1) = \theta + \varphi\big(1 - (1-\varepsilon)^5\big).$$

The one-sided simplex. For $\eta \in \big(0, \frac{a}{1+a}\big]$ let $N_1 = \mathrm{conv}\big(\eta v_1 + (1-\eta)v_2, v_2, v_3, v_4, v_5, v_6\big) \subseteq B$. At its vertices f_1 takes the values $\eta - a(1-\eta) \leq 0$, $-a$, $-(1-a)$, 0, 0, 0, so $N_1 \subseteq \{f_1 \leq 0\}$. By Lemma 4.1(b), $\mathrm{vol}(N_1) = \eta$.

4.2 The fibers and the polytope

Fix rationals $\delta \in (0, 1)$ and $h > 0$ with $h \leq 1 + 2\delta$, put $\eta = h^5/(1+2\delta)^5$, and assume $\eta \leq a/(1+a)$. Let $g = \delta\sigma$, a linear function on $\mathbb{R}^{12}$. The fibers A_z for $z \in \mathcal{Z}$ are listed in Table 1, with K_j, T_j, N_j

the images of K_1, T_1, N_1 under the symmetry (d). Their volumes are

$$\mathrm{vol}(A_z) = \begin{cases} 1 & z = 0, \\ (1+\delta)^5 V & z \text{ paired}, \\ (1-\delta)^5(\theta+\varphi) & z \text{ light}, \\ (1+2\delta)^5\eta = h^5 & z \text{ one-sided or free}. \end{cases}$$

Lemma 4.2. *For every $z \in \mathcal{Z}$:*

(F1) $1 + g(z) > 0$ and $A_z \subseteq (1+g(z))B$;

(F2) if $z \neq 0$ and $\tau_j(z) = 0$, then $A_z \subseteq \{f_j \leq 0\}$;

(F3) if z is heavy, then $|f_j| \leq 1 + 2\delta < 3$ on A_z for $j = 1, 2, 3$.

Proof. (F1) By Table 1, $g(z) = \delta\sigma(z) \geq -\delta > -1$. The fibers of the origin, paired, light and one-sided points are subsets of λB with $\lambda = 1, 1+\delta, 1-\delta, 1+2\delta$, which equals $1 + g(z)$ in each case. For free z we have $hB \subseteq (1+2\delta)B \subseteq (1+g(z))B$, since $0 \in B$ and $g(z) \geq 3\delta$.

(F2) By Table 1, a nonzero z with $\tau_j(z) = 0$ is paired or light of a type $k \neq j$, or one-sided of type j. In the first case A_z is a positive dilate of a subset of T_k, which lies in $\{f_j \leq 0\}$ because $j \neq k$. In the second case $A_z = (1+2\delta)N_j \subseteq \{f_j \leq 0\}$. The conditions are preserved by positive dilations because f_j is linear.

(F3) Heavy fibers lie in $(1+2\delta)B$ by (F1) and the argument for free z; now use $|f_j| \leq 1$ on B and $\delta < 1$. □

Define

$$C = \mathrm{conv} \bigcup_{z \in \mathcal{Z}} \{z\} \times A_z \ \subset \ \mathbb{R}^{12} \times \mathbb{R}^5.$$

Lemma 4.3. *C is a full-dimensional rational polytope containing 0. Its fiber over $z \in \mathbb{Z}^{12}$ is A_z if $z \in \mathcal{Z}$ and empty otherwise. Consequently, for every halfspace H,*

$$\mu(C \cap H) = \sum_{z \in \mathcal{Z}} \mathrm{vol}\big(A_z \cap H_z\big), \qquad \mu(C) = W := 1 + 756(1+\delta)^5 V + 3L + 3339\, h^5,$$

where $L = 252(1-\delta)^5(\theta+\varphi)$ is the total volume of the light fibers of one type, and $3339 = 630 + 2709$ counts the one-sided and free points.

Proof. The sets T_j, K_j are cut out of B by rational inequalities, N_j has rational vertices, and all dilation factors are rational. So C is the convex hull of finitely many rational points. It contains $\{0\} \times B$, and the points $(\mathbf{1}_{\{i\}}, 0)$, $i \in [10]$, and $(0, -e_1), (0, -e_2)$ lie in $\mathcal{Z}$ (they are free, respectively one-sided with $J = \emptyset$), so C is full-dimensional.

Let $(z, x) \in C$ with $z \in \mathbb{Z}^{12}$. Write $(z, x) = \sum_m \lambda_m (z_m, x_m)$ with $\lambda_m > 0$, $\sum_m \lambda_m = 1$, $z_m \in \mathcal{Z}$ and $x_m \in A_{z_m}$. If $z \neq 0$, then Lemma 3.2 gives $z_m = z$ for all m, so $z \in \mathcal{Z}$ and $x \in A_z$ by convexity of A_z. If $z = 0$, then $\sum_m \lambda_m (1 + g(z_m)) = 1 + g(0) = 1$ by linearity of g, so

$$x = \sum_m \lambda_m \big(1 + g(z_m)\big) \cdot \frac{x_m}{1 + g(z_m)}$$

is a convex combination of points of B by (F1), and $x \in B = A_0$. The reverse inclusions are clear. □

5 Covering by halfspaces

5.1 The origin and the light fibers

Lemma 5.1. *For $j = 1, 2, 3$, the set $H_j = \{(z, x) \in \mathbb{R}^{12} \times \mathbb{R}^5 : f_j(x) \geq 3\tau_j(z)\}$ is a halfspace with $\mu(C \cap H_j) = \beta + L$. Every point of $\{0\} \times B$ and of $\{z\} \times A_z$ with z light lies in $H_1 \cup H_2 \cup H_3$.*

Proof. H_j is defined by the nonzero linear functional $(z, x) \mapsto f_j(x) - 3\tau_j(z)$. We compute $\mathrm{vol}(A_z \cap \{f_j \geq 3\tau_j(z)\})$ for each $z \in \mathcal{Z}$ and apply Lemma 4.3; see Fig. 2(a). For $z = 0$ it is $\mathrm{vol}(B \cap \{f_j \geq 0\}) = \beta$. If z is light of type j, then $\tau_j(z) = -1$ and $f_j \geq -1 > -3$ on $A_z \subseteq B$, so the whole fiber counts. These fibers have total volume L. If $z \neq 0$ and $\tau_j(z) = 0$, the trace lies in $A_z \cap \{f_j = 0\}$ by (F2), which has volume 0. In all remaining cases $\tau_j(z) \geq 1$ by (4), so z is heavy and the trace is empty by (F3). The covering statement follows from property (b) of the f_j and from the case of light points. □

5.2 Tilted caps on heavy fibers

Lemma 5.2. *Let $z \in \mathcal{Z} \setminus \{0\}$, let ψ be a nonconstant affine function on $\mathbb{R}^5$, and let $\kappa \in \mathbb{R}$. There is a halfspace H with $H_z = \{\psi \geq \kappa\}$ and $\mu(C \cap H) = \mathrm{vol}(A_z \cap \{\psi \geq \kappa\})$.*

Proof. For $\rho > 0$ let $H = \{(z', x') : \psi(x') - \psi(0) + \rho\,\ell_z(z') \geq \kappa - \psi(0) + \rho\,\ell_z(z)\}$. The left-hand side is a nonzero linear functional, and $H_z = \{\psi \geq \kappa\}$. For $z' \in \mathcal{Z} \setminus \{z\}$ we have $\ell_z(z') < \ell_z(z)$. Since ψ is bounded on the finitely many compact fibers $A_{z'}$, the halfspace H misses all of them once ρ is large. Now apply Lemma 4.3. □

Corollary 5.3. *Let $M = (5h/6)^5$. Every point of $\{z\} \times A_z$ with z free or one-sided lies in a halfspace of mass M.*

Proof. A_z is a simplex of volume h^5. A point $x \in A_z$ has some barycentric coordinate $\lambda_i(x) \geq 1/6$ with respect to A_z, and $A_z \cap \{\lambda_i \geq 1/6\}$ is a homothetic copy of A_z with ratio $5/6$, of volume $(5/6)^5 h^5 = M$. Apply Lemma 5.2 with $\psi = \lambda_i$ and $\kappa = 1/6$. □

5.3 The paired fibers

A paired fiber $(1 + \delta)K_j$ is not a simplex, but it is covered by the halfspaces that cut caps from the simplex $(1 + \delta)U_j$. Fix positive rationals $t_0, \ldots, t_5$ with $\sum_i t_i = 1$ and $\frac{1}{7} < t_i < \frac{1}{5}$. Using the thresholds $t_i = \frac{1}{6}$, as for a simplex, would make the caps unbalanced: the slab enlarges every cap except the one at w_3 (index 2), so we take t_2 slightly smaller than the others.

Lemma 5.4. *Every point of $\{z\} \times A_z$ with z paired lies in a halfspace of mass $(1 + \delta)^5 V_i$ for some $i \in \{0, \ldots, 5\}$, where $V_2 = \theta(1 - t_2)^5$ and, for $i \neq 2$,*

$$V_i = \theta(1 - t_i)^5 + \varphi\,\frac{(1 - t_i)^5 - \big(1 - t_i - (1 - q_i)\varepsilon\big)^5}{1 - q_i}.$$

Proof. By symmetry let z be of type 1 and $x \in (1 + \delta)K_1$. Since $\sum_i \nu_i = 1 = \sum_i t_i$ on $\mathbb{R}^5$, some i has $\nu_i(x/(1+\delta)) \geq t_i$. Applying Lemma 5.2 with $\psi(x') = \nu_i(x'/(1+\delta))$ and $\kappa = t_i$ gives a halfspace of mass $(1 + \delta)^5 V_i$ with $V_i = \mathrm{vol}(K_1 \cap \{\nu_i \geq t_i\})$. It remains to compute V_i, separately on U_1 and on the slab $P_1 \cap \{p \leq \varepsilon\}$, which overlap in a null set.

On U_1, the set $\{\nu_i \geq t_i\}$ is a homothetic copy of U_1 with ratio $1 - t_i$, of volume $\theta(1 - t_i)^5$.

On the slab, $\nu_2 = q_2 p \leq 0 < t_2$ by (5), so for $i = 2$ there is no contribution. Let $i \neq 2$. Use p and the four coordinates p_k, $k \notin \{i, 2\}$, as affine coordinates on P_1. In these coordinates P_1 is a standard

simplex, of coordinate volume $1/120$, so volume equals 120φ times coordinate volume. Substituting $p_i = 1 - p - \sum_{k\neq i,2} p_k$ into (5), the condition $\nu_i \geq t_i$ becomes $\sum_{k\neq i,2} p_k \leq 1 - t_i - (1-q_i)p$. This also implies $p_i \geq t_i - q_i p > 0$, since $q_i < 1$ and $q_i p \leq \varepsilon < t_i$. For fixed $p \in [0,\varepsilon]$, the four coordinates therefore range over a simplex of coordinate volume $(1-t_i-(1-q_i)p)^4/24$, which is positive because $\varepsilon < 10^{-3}$ and $t_i < \frac{1}{5}$. Hence the slab contributes

$$120\varphi \int_0^{\varepsilon} \frac{\big(1-t_i-(1-q_i)p\big)^4}{24}\,dp = \varphi\,\frac{(1-t_i)^5 - \big(1-t_i-(1-q_i)\varepsilon\big)^5}{1-q_i}. \qquad \square$$

6 Parameters and proof of the theorem

By Lemmas 5.1 and 5.4 and Corollary 5.3, the construction works as soon as the following strict inequalities hold:

$$\beta + L < M, \qquad (1+\delta)^5 V_i < M \quad (i = 0,\dots,5), \qquad 4096\,h^5 < W. \tag{6}$$

The first bounds the halfspaces through the origin fiber, the second the caps on paired fibers, and the third is the target (1). In view of the formula for W in Lemma 4.3, the third inequality is exactly the account (2):

$$W - 4096\,h^5 = (1 + 3L - h^5) - 756\big(h^5 - (1+\delta)^5 V\big).$$

We take

$$a = \frac{663}{1000}, \qquad \delta = \frac{664201}{10^6}, \qquad \varepsilon = \frac{13}{20000}, \qquad h = \frac{1111946415}{10^9},$$
$$(t_0,\dots,t_5) = 10^{-9}(166735371,\ 166735294,\ 166324902,\ 166734811,\ 166734811,\ 166734811).$$

These satisfy all side conditions: $\frac{1}{2} < a < 1$, $0 < \delta < 1$, $0 < \varepsilon < 10^{-3}$, $h \leq 1+2\delta$, $\eta = h^5/(1+2\delta)^5 \approx 0.0248 \leq a/(1+a)$, $\sum_i t_i = 1$ and $\frac{1}{7} < t_i < \frac{1}{5}$. The derived constants are

$$\theta = \tfrac{221}{1663}, \qquad \varphi = \tfrac{337}{4011}, \qquad \beta = \tfrac{1000000}{2223431}, \qquad \tfrac{5h}{6} = \tfrac{74129761}{80000000},$$
$$q_0 = \tfrac{776569}{886431}, \qquad q_1 = \tfrac{1011}{1337}, \qquad q_2 = -\tfrac{560431}{886431}.$$

Exact evaluation in rational arithmetic gives the values in Table 2, and in particular

$$M - \beta - L > 10^{-6}, \qquad M - \max_i (1+\delta)^5 V_i > 10^{-9}, \qquad W - 4096\,h^5 > 4\cdot 10^{-5},$$

which proves (6). These parameters are not canonical; any choice satisfying the side conditions and (6) works.

Proof of Theorem 1.2. By Lemma 4.3, C is a full-dimensional rational polytope with $\mu(C) = W$, a finite sum of positive volumes. Let $(z,x) \in \mathbb{Z}^{12} \times \mathbb{R}^5$. If $(z,x) \notin C$, strict separation gives a halfspace containing (z,x) and disjoint from C, of mass 0. Otherwise $z \in \mathcal{Z}$ and $x \in A_z$ by Lemma 4.3, and (z,x) lies in a halfspace of mass at most M:

- if $z = 0$ or z is light, in some H_j, of mass $\beta + L < M$ (Lemma 5.1);
- if z is free or one-sided, in a tilted cap of mass M (Corollary 5.3);
- if z is paired, in a tilted cap of mass $(1+\delta)^5 V_i < M$ (Lemma 5.4).

All halfspaces are closed, so boundary points of fibers need no separate treatment. Let $r = M/W = (5/6)^5 h^5/W$, which is rational. Every point of $\mathbb{Z}^{12}\times\mathbb{R}^5$ lies in a halfspace H with $\mu(C\cap H) \leq r\,\mu(C)$, and $W > 4096\,h^5$ gives $r < (5/6)^5/2^{12}$. $\square$

quantity	$\lfloor 10^{12}\cdot\rfloor$	margin	$\lfloor 10^{12}\cdot\rfloor$
h^5	1 699 884 034 250	$M-\beta-L$	1 262 439
M	683 145 268 393	$M-(1+\delta)^5V_0$	7 542
β	449 755 355 574	$M-(1+\delta)^5V_1$	6 288
L	233 388 650 379	$M-(1+\delta)^5V_2$	1 211
$3L$	700 165 951 139	$M-(1+\delta)^5V_i$, $i=3,4,5$	7 869
$(1+\delta)^5\theta$	1 696 402 554 056	$h^5-(1+\delta)^5V$	312 774
$(1+\delta)^5V$	1 699 883 721 475	$1+3L-h^5$	281 916 889
W	6 962 725 049 747 587	$W-4096\,h^5$	45 459 473

Table 2: Exact values (rounded down after multiplying by 10^{12}) of the relevant quantities for the chosen parameters, in units where $\mathrm{vol}(B)=1$.

AI disclosure. The counterexample of Theorem 1.2 and its proof were obtained by GPT-6 Astra [8] in Codex, using a fact graph system similar to Danus [6]. The resulting graph, available at https://github.com/Hongyu-Cheng/weak-oertel-conjecture, contains 3503 facts, and the counterexample is a single fact in it whose proof uses no other fact. The authors rewrote this proof as the argument of Sections 2–6, checked it with GPT-6.1 Sol [9], and take full responsibility for the paper.

A Computer verification

Two of the exact rational computations, the census of $\mathcal{Z}$ in Lemma 3.1 and the inequalities (6), are checked by a short Python script using exact rational arithmetic. The script generates the points of $\mathcal{Z}$ and confirms their numbers, counting the one-sided points of the three types together. It asserts the three margins of Section 6, which imply (6), and the inequalities among the parameters that Sections 4 and 5 use, three of them ($\frac{1}{2}<a$, $\varepsilon<10^{-3}$ and $\frac{1}{7}<t_i<\frac{1}{5}$) only through the consequences that Lemma 5.4 needs. The script does not prove that $\mathcal{Z}$ contains all integer points of $\mathrm{conv}(\mathcal{Z})$; that is Lemma 3.2.

The bound on the depth in Theorem 1.2 has also been formalized in Lean 4 with mathlib, including the consequences of Lemma 3.2. The final statement reads as follows. Here `Coordinate` is `Fin 10 ⊕ Fin 2`, `IntegerSpace` is `Coordinate → ℝ`, `E` is `Fin 5 → ℝ`, `Ambient` is `IntegerSpace × E`, and `mixedIntegerMass D` is the extended nonnegative sum, over all `w : Coordinate → ℤ`, of the Lebesgue volume of the fiber of `D` over `w`.

```
theorem sharp_oc_counterexample :
    ∃ C : Set Ambient, IsCompact C ∧ Convex ℝ C ∧ (interior C).Nonempty ∧
      ((0 : IntegerSpace), (0 : E)) ∈ C ∧
      0 < mixedIntegerMass C ∧ mixedIntegerMass C ≠ ⊤ ∧
      ∃ r : ℝ, 0 < r ∧ r < (((5 : ℝ) / 6)^5 / (2 : ℝ)^12) ∧
        ∀ (z : Coordinate → ℤ) (x : E),
          ∃ F : Ambient →L[ℝ] ℝ, ∃ b : ℝ, F ≠ 0 ∧ b ≤ F ((fun j => (z j : ℝ)), x) ∧
            (mixedIntegerMass (C ∩ {p | b ≤ F p})).toReal /
              (mixedIntegerMass C).toReal ≤ r
```

Rationality of C and r is not part of this statement; it follows from Lemma 4.3 and $r=M/W$.

The development uses Lean toolchain `leanprover/lean4:v4.34.1` and mathlib revision `d13f23b7`. The final theorem depends only on the axioms `propext`, `Classical.choice` and `Quot.sound`. The repository https://github.com/Hongyu-Cheng/weak-oertel-conjecture holds the script as `research/sharp-dimension-17/check.py` and the Lean development in the folder `lean`. The final theorem,

`sharp_oc_counterexample` in the namespace `SharpOC.Selector17`, is proved in the file `lean/SharpOC/Selector17Counterexample.lean`. With Python 3 and `elan` installed, the following commands, run from the root of the repository, rerun both checks. The last command fails unless the final theorem and the other results listed in `lean/CheckAxioms.lean` depend only on these three axioms.

```
python3 research/sharp-dimension-17/check.py
cd lean && lake exe cache get && lake build && lake env lean CheckAxioms.lean
```

References

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