553.665 Introduction to Convexity, Fall 2022
Section 11

Hongyu Cheng

December 2, 2022

Definition 1 (Dual cone)**.** Let K be a convex cone, the set $K^* = \{\mathbf{y} : \langle \mathbf{x}, \mathbf{y} \rangle \geq 0 \text{ for all } \mathbf{x} \in K\}$ is called the *dual cone* of K.

Exercise 1 (Positive semidefinite cone)**.** We use $\mathcal{S}^n$ to denote all the symmetric $n \times n$ matrices, and all the $n \times n$ positive semidefinite matrices will be denoted by $\mathcal{S}^n_+$. On the set of $\mathcal{S}^n$, we use the standard inner product $\operatorname{tr}(AB) = \sum_{i,j=1}^n A_{ij} B_{ij}$. Prove that $\mathcal{S}^n_+$ is a closed convex cone and self-dual.

Proof. First we prove that $\mathcal{S}^n_+$ is a closed convex cone. Observe that

$$\begin{aligned}
\mathcal{S}^n_+ &= \{A \in \mathcal{S}^n : \mathbf{x}^T A \mathbf{x} \geq 0 \text{ for all } \mathbf{x} \in \mathbb{R}^n\} \\
&= \bigcap_{\mathbf{x} \in \mathbb{R}^n} \{A \in \mathcal{S}^n : \operatorname{tr}(\mathbf{x}^T A \mathbf{x}) \geq 0\} \\
&= \bigcap_{\mathbf{x} \in \mathbb{R}^n} \{A \in \mathcal{S}^n : \operatorname{tr}(\mathbf{x}\mathbf{x}^T A) \geq 0\} \\
&= \bigcap_{\mathbf{x} \in \mathbb{R}^n} \{A \in \mathcal{S}^n : \langle \mathbf{x}\mathbf{x}^T, A \rangle \geq 0\}
\end{aligned}$$

is an intersection of closed halfspaces, which implies $\mathcal{S}^n_+$ is a closed convex set. To show it is a cone, we take any $A, B \in \mathcal{S}^n_+$ and $\lambda, \gamma \geq 0$, then for any $\mathbf{x} \in \mathbb{R}^n$ we have

$$\mathbf{x}^T(\lambda A + \gamma B)\mathbf{x} = \lambda(\mathbf{x}^T A \mathbf{x}) + \gamma(\mathbf{x}^T B \mathbf{x}) \geq 0,$$

so $\lambda A + \gamma B \in \mathcal{S}^n_+$.

Then we prove that $\mathcal{S}^n_+$ is self-dual, i.e., for $A, B \in \mathcal{S}^n$, $\operatorname{tr}(AB) \geq 0,\ \forall A \in \mathcal{S}^n_+ \iff B \in \mathcal{S}^n_+$. Suppose $B \notin \mathcal{S}^n_+$, then there exists $\mathbf{x} \in \mathbb{R}^n$ with

$$\mathbf{x}^T B \mathbf{x} = \operatorname{tr}(\mathbf{x}\mathbf{x}^T B) < 0.$$

Hence the positive semidefinite matrix $A = \mathbf{x}\mathbf{x}^T$ satisfies $\operatorname{tr}(AB) < 0$, which implies $B \notin (\mathcal{S}^n_+)^*$.

Now suppose $A, B \in \mathcal{S}^n_+$. We can express A in terms of its eigenvalue decomposition as $A = \sum_{i=1}^n \lambda_i \mathbf{x}_i \mathbf{x}_i^T$, where the eigenvalues $\lambda_i \geq 0,\ i = 1, \ldots, n$. Then we have

$$\operatorname{tr}(BA) = \operatorname{tr}\left(B \sum_{i=1}^n \lambda_i \mathbf{x}_i \mathbf{x}_i^T\right) = \sum_{i=1}^n \lambda_i \mathbf{x}_i^T B \mathbf{x}_i \geq 0.$$

This shows that $B \in (\mathcal{S}^n_+)^*$. ◻

Exercise 2 (Dual of a norm cone [1])**.** Let $\|\cdot\|$ be a norm on $\mathbb{R}^n$. The dual of the associated cone $K = \{(\mathbf{x}, t) \in \mathbb{R}^{n+1} : \|\mathbf{x}\| \le t\}$ is the cone defined by the dual norm, i.e.,

$$K^* = \{(\mathbf{u}, v) \in \mathbb{R}^{n+1} : \|\mathbf{u}\|_* \le v\},$$

where the dual norm is given by $\|\mathbf{u}\|_* = \sup\{\langle \mathbf{u}, \mathbf{x}\rangle : \|\mathbf{x}\| \le 1\}$.

Proof. To prove the result we need to show that

$$\langle \mathbf{x}, \mathbf{u}\rangle + tv \ge 0 \text{ whenever } \|\mathbf{x}\| \le t \iff \|\mathbf{u}\|_* \le v.$$

"$\Longrightarrow$" : Suppose to the contrary that $\|\mathbf{u}\|_* > v$, then by the definition of the dual norm, there exists an $\mathbf{x}$ with $\|\mathbf{x}\| \le 1$ and $\langle \mathbf{x}, \mathbf{u}\rangle > v$. Taking $t = 1$, we have $\langle \mathbf{u}, -\mathbf{x}\rangle + v < 0$, which leads to a contradiction.
"$\Longleftarrow$" : Suppose $\|\mathbf{u}\|_* \le v$ and $\|\mathbf{x}\| \le t$ for some $t > 0$. Applying the definition of the dual norm, and the fact that $\|-\mathbf{x}/t\| \le 1$, we have $\langle \mathbf{u}, -\mathbf{x}/t\rangle \le \|\mathbf{u}\|_* \le v$, and therefore $\langle \mathbf{u}, \mathbf{x}\rangle + vt \ge 0$. □

Exercise 3 (Second-order conditions for K-convexity [1])**.** Let $K \subseteq \mathbb{R}^m$ be a closed, convex, pointed cone, with associated generalized inequality $\preceq_K$. Show that a twice differentiable function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$, with convex domain, is K-convex if and only if for all $\mathbf{x} \in \operatorname{dom} \mathbf{f}$ and all $\mathbf{y} \in \mathbb{R}^n$,

$$0 \preceq_K \sum_{i,j=1}^{n} \frac{\partial^2 \mathbf{f}(\mathbf{x})}{\partial \mathbf{x}_i \partial \mathbf{x}_j} \mathbf{y}_i \mathbf{y}_j.$$

(Here $\partial^2 \mathbf{f}/\partial \mathbf{x}_i \partial \mathbf{x}_j \in \mathbb{R}^m$, with components $\partial^2 \mathbf{f}_k/\partial \mathbf{x}_i \partial \mathbf{x}_j$, for $k = 1, \dots, m$.)

Proof. It's not hard to show $\mathbf{f}$ is K-convex if and only if $\langle \mathbf{c}, \mathbf{f}\rangle$ is convex for all $\mathbf{c} \in K^*$ (similar to Proposition 9.2.9 in lecture notes). The Hessian of $\langle \mathbf{c}, \mathbf{f}(\mathbf{x})\rangle$ is

$$\nabla^2(\langle \mathbf{c}, \mathbf{f}(\mathbf{x})\rangle) = \sum_{k=1}^{n} \mathbf{c}_k \nabla^2 \mathbf{f}_k(\mathbf{x}).$$

This is positive semidefinite if and only if for all $\mathbf{y}$,

$$\mathbf{y}^T \nabla^2(\langle \mathbf{c}, \mathbf{f}(x)\rangle)\mathbf{y} = \sum_{i,j=1}^{n} \sum_{k=1}^{n} \mathbf{c}_k \nabla^2 \mathbf{f}_k(\mathbf{x}) \mathbf{y}_i \mathbf{y}_j = \sum_{k=1}^{n} \mathbf{c}_k \left(\sum_{i,j=1}^{n} \nabla^2 \mathbf{f}_k(\mathbf{x}) \mathbf{y}_i \mathbf{y}_j \right) \ge 0,$$

which by definition of dual cone is equivalent to

$$\sum_{i,j=1}^{n} \nabla^2 \mathbf{f}_k(\mathbf{x}) \mathbf{y}_i \mathbf{y}_j \in K.$$

□

References

[1] Stephen Boyd, Stephen P Boyd, and Lieven Vandenberghe. Convex optimization. Cambridge university press, 2004.