553.665 Introduction to Convexity, Fall 2022
Section 3

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1 Review

Question 1. Let $C \subseteq \mathbb{R}^d$ be a closed convex set, is every face of C an exposed face?
Answer. No. In $\mathbb{R}^2$, let $X = \operatorname{conv}(\{(-2,0),(0,0),(0,2),(-2,2)\})$, $Y = \{(x,y) : x^2 + (y-1)^2 \leq 1\}$, then $X \cup Y$ has a face $\{(0,0)\}$ which is not an exposed face.

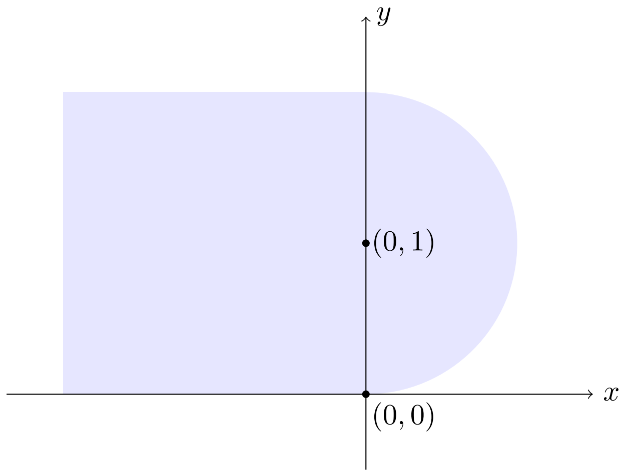


Figure 1: A counterexample for Question 1.

Question 2. Let $X,\ Y \subseteq \mathbb{R}^d$ be closed sets, is $X + Y$ a closed set?
Answer. It's not always true. Some counterexamples:

1. $X = \{n + 2^{-n} : n \in \mathbb{N}_+\}$, $Y = \mathbb{Z}$ are closed sets in $\mathbb{R}$. Observe that a convergent sequence $\{2^{-n}\}_{n=1}^{+\infty} \subseteq X + Y$, but $\lim_{n\to+\infty} 2^{-n} = 0 \notin X + Y$.
2. $X = \{(x,y) : xy \geq 1\}$, $Y = \{(0,y) : y \in \mathbb{R}\}$ are closed sets in $\mathbb{R}^2$. However, $X + Y = \{(x,y) : x \neq 0\}$ is not closed.

Question 3. Let $X \subseteq \mathbb{R}^d$ be a convex set, is every face of X a closed set?
Answer. It's not always true. Let X be an open unit ball in $\mathbb{R}^d$, then a trivial face is X itself, which is not a closed set.

2 Exercises

Exercise 1. Show that if X is compact and Y is closed, then $X+Y$ is closed.

Proof. Consider any convergent sequence $\mathbf{x}^i + \mathbf{y}^i \to \mathbf{z}$ such that $\mathbf{x}^i \in X,\ \mathbf{y}^i \in Y$ for all $i \in \mathbb{N}_+$. We need to show that $\mathbf{z} \in X+Y$. Since X is compact, it has a convergent subsequence, and there is some $\mathbf{x} \in X$ such that $\mathbf{x}^{i_k} \to \mathbf{x}$. Along this subsequence, we have that

$$\lim_{k \to +\infty} (\mathbf{x}^{i_k} + \mathbf{y}^{i_k}) = \mathbf{z},$$

so $\mathbf{y}^{i_k} \to \mathbf{z} - \mathbf{x}$ as $k \to +\infty$. Since Y is closed, $\mathbf{z} - \mathbf{x} \in Y$, so we get $\mathbf{z} = \mathbf{x} + (\mathbf{z} - \mathbf{x}) \in X + Y$. □

Exercise 2 (Equivalent definition of face)**.** Let $C \subseteq \mathbb{R}^d$ be a convex set. Then a convex subset $F \subseteq C$ is a face if and only if $\forall \mathbf{z} \in F$, $\mathbf{z} = \lambda \mathbf{x}^1 + (1-\lambda)\mathbf{x}^2$ for some $\mathbf{x}^1, \mathbf{x}^2 \in C$ and $\lambda \in (0,1)$ implies $\mathbf{x}^1, \mathbf{x}^2 \in F$.

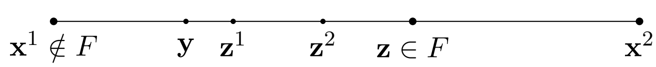
Proof. ($\Longleftarrow$) This is a trivial case by taking $\lambda = \frac{1}{2}$.

($\Longrightarrow$) Consider any $\mathbf{x}^1, \mathbf{x}^2 \in C,\ \lambda \in (0,1)$ with $\mathbf{z} = \lambda \mathbf{x}^1 + (1-\lambda)\mathbf{x}^2 \in F$, and WLOG we suppose $\mathbf{x}^1 \in C \backslash F$. Now if $\mathbf{z}^1 = (\mathbf{x}^1 + \mathbf{z})/2 \in F$, we would have a contradiction, so we may construct $\mathbf{z}^k = (\mathbf{z}^{k-1} + \mathbf{z})/2 \in C \backslash F$ for all $k \geq 1$. Then if for some $\lambda' \in (\lambda, 1)$, $\mathbf{y} = \lambda' \mathbf{x}^1 + (1-\lambda')\mathbf{x}^2 \in F$, the line segment connecting $\mathbf{y}$ and $\mathbf{z}$ would have to lie in F by convexity, and this would contain $\mathbf{z}^k$ for some $k \geq 1$. Thus $\{\lambda' \mathbf{x}^1 + (1-\lambda')\mathbf{x}^2 : \lambda' \in (\lambda, 1)\} \subseteq C \backslash F$. However,

$$\mathbf{z} = \frac{\lambda_1 \mathbf{x}^1 + (1-\lambda_1)\,\mathbf{x}^2}{2} + \frac{\lambda_2 \mathbf{x}^1 + (1-\lambda_2)\,\mathbf{x}^2}{2} \in F,$$

where $\lambda_1 = \lambda - \varepsilon,\ \lambda_2 = \lambda + \varepsilon$ with $\varepsilon = \frac{1}{2}\min\{\lambda, 1-\lambda\}$. This leads to a contradiction since the second vector is not in F.

□



Hongyu: I stopped here in today's session.

Exercise 3. Let $C \subseteq \mathbb{R}^d$ be a convex set and let $X \subseteq C$ be convex. Let $\mathbf{x} \in \operatorname{relint}(X)$. Suppose F is a face of C such that $\mathbf{x} \in F$. Prove that $X \subseteq F$.

Proof. $\forall \mathbf{y} \in X$, observe that $2\mathbf{x} - \mathbf{y} \in \operatorname{aff}(X)$ since $\mathbf{x},\ \mathbf{y} \in X$ and $2 + (-1) = 1$. Since $\mathbf{x} \in \operatorname{relint}(X)$, there is some $\varepsilon > 0$ such that $\mathbf{z} = \mathbf{x} + \varepsilon((2\mathbf{x} - \mathbf{y}) - \mathbf{x}) \in X$. Therefore,

$$\mathbf{x} = \frac{1}{1+\varepsilon}\mathbf{z} + \frac{\varepsilon}{1+\varepsilon}\mathbf{y},$$

so $\mathbf{y} \in F$, since F is a face. This proves that $X \subseteq F$. □

Exercise 4. Let $C \subseteq \mathbb{R}^d$ be a compact, convex set. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a linear function given by $f(\mathbf{y}) = \langle \mathbf{a}, \mathbf{y} \rangle$ for some $\mathbf{a} \in \mathbb{R}^d$. Show that there exists an extreme point $\mathbf{v} \in C$ such that $f(\mathbf{v}) \leq f(\mathbf{x})$ for every $\mathbf{x} \in C$.

Proof. f is a continuous function by definition, then Weierstrass theorem yields that there is some point $\mathbf{u} \in C$ such that $f(\mathbf{u}) \leq f(\mathbf{x})$ for all $\mathbf{x} \in C$. Since C is compact and convex, by Krein-Milman $C =$

conv (ext(C)). Then for some $\mathbf{v}^i \in \text{ext}(C)$ and $\lambda_i \geq 0$ summing to 1, we have $\mathbf{u} = \sum_{i=1}^k \lambda_i \mathbf{v}^i$ and

$$\langle \mathbf{a}, \mathbf{u} \rangle = \sum_{i=1}^{k} \lambda_i \left\langle \mathbf{a}, \mathbf{v}^i \right\rangle \geq \sum_{i=1}^{k} \lambda_i \langle \mathbf{a}, \mathbf{u} \rangle = \langle \mathbf{a}, \mathbf{u} \rangle.$$

Thus the inequality must be satisfied with equality, and $\langle \mathbf{a}, \mathbf{v}^i \rangle = \langle \mathbf{a}, \mathbf{u} \rangle$ for each of these $\mathbf{v}^i$. □

Exercise 5. Show that if D is a closed convex cone, then any face of D is a convex cone.

Proof. Let $F \subseteq D$ be a face. We first want to show that $0 \in F$. Let $\mathbf{x} \in F$ and $\mu > 1$, then

$$\mathbf{x} = \frac{1}{\mu}(\mu \mathbf{x}) + \frac{\mu - 1}{\mu} 0,$$

so $\mu \mathbf{x}, 0 \in D$ since F is a face.

Since F is convex, then we just need to show that $\forall \mathbf{x} \in F,\ \lambda \geq 0$, we have that $\lambda \mathbf{x} \in F$. We have shown the case when $\lambda > 1$ when arguing that $0 \in F$. Now suppose $\lambda \in (0, 1]$, then

$$\lambda \mathbf{x} = \lambda \underbrace{\mathbf{x}}_{\in F} + (1 - \lambda) \underbrace{0}_{\in F} \in F$$

by convexity, which completes the proof. □