553.665 Introduction to Convexity, Fall 2022

Section 4

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Exercise 1. Let P_1, P_2 be two polytopes in $\mathbb{R}^d$. Show that $P_1 + P_2$ is a polytope.

Proof. By Minkowski-Weyl theorem, $P_1 = \operatorname{conv}\left(\{\mathbf{u}^1, \ldots, \mathbf{u}^m\}\right), P_2 = \operatorname{conv}\left(\{\mathbf{v}^1, \ldots, \mathbf{v}^n\}\right)$ for some $\mathbf{u}^i, \mathbf{v}^j \in \mathbb{R}^d$. To prove $P_1 + P_2$ is a polytope, we prove that

$$P_1 + P_2 = \operatorname{conv}\left(\{\mathbf{u}^i + \mathbf{v}^j\}_{(i,j)\in\{1,\ldots,m\}\times\{1,\ldots,n\}}\right).$$

$\forall \mathbf{x} \in P_1 + P_2$, $\mathbf{x} = \mathbf{u} + \mathbf{v}$ for some $\mathbf{u} \in P_1, \mathbf{v} \in P_2$. Then by convexity, there exist some $\alpha_i, \beta_j \geq 0$ with $\sum_{i=1}^m \alpha_i = 1, \sum_{j=1}^n \beta_j = 1$ such that $\mathbf{u} = \sum_{i=1}^m \alpha_i \mathbf{u}^i,\ \mathbf{v} = \sum_{j=1}^n \beta_j \mathbf{v}^j$. Therefore,

$$\begin{aligned}
\mathbf{x} = \mathbf{u} + \mathbf{v} &= \sum_{i=1}^m \alpha_i \mathbf{u}^i + \sum_{j=1}^n \beta_j \mathbf{v}^j \\
&= \sum_{i=1}^m \alpha_i \mathbf{u}^i \underbrace{\sum_{j=1}^n \beta_j}_{=1} + \underbrace{\sum_{i=1}^m \alpha_i}_{=1} \sum_{j=1}^n \beta_j \mathbf{v}^j \\
&= \sum_{i=1}^m \left(\alpha_i \mathbf{u}^i \sum_{j=1}^n \beta_j + \alpha_i \sum_{j=1}^n \beta \mathbf{v}^j \right) \\
&= \sum_{i=1}^m \sum_{j=1}^n \alpha_i \beta_j \left(\mathbf{u}^i + \mathbf{v}^j\right).
\end{aligned}$$

Observe that $\sum_{i=1}^m \sum_{j=1}^n \alpha_i\beta_j = \sum_{i=1}^m \alpha_i \sum_{j=1}^n \beta_j = 1$ with $\alpha_i\beta_j \geq 0$, then $\mathbf{x}$ is a convex combination of points in $\{\mathbf{u}^i + \mathbf{v}^j\}_{(i,j)\in\{1,\ldots,m\}\times\{1,\ldots,n\}}$, so $\mathbf{x} \in \operatorname{conv}\left(\{\mathbf{u}^i + \mathbf{v}^j\}_{(i,j)\in\{1,\ldots,m\}\times\{1,\ldots,n\}}\right)$.

To show the reverse inclusion, $\forall \mathbf{x} \in \operatorname{conv}\left(\{\mathbf{u}^i + \mathbf{v}^j\}_{(i,j)\in\{1,\ldots,m\}\times\{1,\ldots,n\}}\right)$, there exist $\lambda_{ij} \geq 0$ with $\sum_{i=1}^m \sum_{j=1}^n = 1$ such that $\mathbf{x} = \sum_{i=1}^m \sum_{j=1}^n \lambda_{ij} \left(\mathbf{u}^i + \mathbf{v}^j\right)$.

Then

$$\begin{aligned}
\mathbf{x} &= \sum_{i=1}^{m}\sum_{j=1}^{n}\lambda_{ij}\left(\mathbf{u}^i+\mathbf{v}^j\right)\\
&= \sum_{i=1}^{m}\sum_{j=1}^{n}\lambda_{ij}\mathbf{u}^i+\sum_{j=1}^{n}\sum_{i=1}^{m}\lambda_{ij}\mathbf{v}^j\\
&= \underbrace{\sum_{i=1}^{m}\alpha_i\mathbf{u}^i}_{\in P_1}+\underbrace{\sum_{j=1}^{n}\beta_j\mathbf{v}^j}_{\in P_2}\in P_1+P_2,
\end{aligned}$$

where $\alpha_i=\sum_{j=1}^{n}\lambda_{ij}\ge 0,\ i=1,\dots,m$ and $\beta_j=\sum_{j=1}^{m}\lambda_{ij}\ge 0,\ j=1,\dots,n$ with

$$\sum_{i=1}^{m}\alpha_i=\sum_{j=1}^{n}\beta_j=\sum_{i=1}^{m}\sum_{j=1}^{n}\lambda_{ij}=1.$$

Minkowski-Weyl theorem implies that P_1+P_2 is a polytope since it's a convex hull of finitely many points. □

Exercise 2. Let $A\in\mathbb{R}^{m\times d},\mathbf{b}\in\mathbb{R}^m$. Consider the polyhedron $P=\{\mathbf{x}\in\mathbb{R}^d : A\mathbf{x}\le\mathbf{b}\}$. Show that

$$\operatorname{rec}(P)=\{\mathbf{x}\in\mathbb{R}^d : A\mathbf{x}\le 0\},\quad \operatorname{lin}(P)=\{\mathbf{x}\in\mathbb{R}^d : A\mathbf{x}=0\}.$$

Proof. $\forall\mathbf{w}\in\{\mathbf{x}\in\mathbb{R}^d : A\mathbf{x}\le 0\}$. For any $\lambda\ge 0$ and $\mathbf{x}\in P$,

$$A(\mathbf{x}+\lambda\mathbf{w})=A\mathbf{x}+\lambda A\mathbf{w}\le\mathbf{b}+\lambda 0=\mathbf{b},$$

so $\mathbf{w}\in\operatorname{rec}(P)$. This proves $\{\mathbf{x}\in\mathbb{R}^d : A\mathbf{x}\le 0\}\subseteq\operatorname{rec}(P)$.

To show reverse inclusion, $\forall\mathbf{y}\notin\{\mathbf{x}\in\mathbb{R}^d : A\mathbf{x}\le 0\}$, there exists $i\in\{1,\dots,m\}$ such that

$$\alpha=(A\mathbf{y})_i>0.$$

Suppose that $\mathbf{x}\in P$, and let $\beta=(A\mathbf{x})_i$. Consider $\lambda=(\mathbf{b}_i+1-\beta)/\alpha$, then $\lambda>0$ since $\beta\le\mathbf{b}_i$. Therefore,

$$\begin{aligned}
(A(\mathbf{x}+\lambda\mathbf{y}))_i &= (A\mathbf{x})_i+\lambda(A\mathbf{y})_i\\
&= \beta+\lambda\alpha\\
&= \beta+(\mathbf{b}_i+1-\beta)\\
&= \mathbf{b}_i+1>\mathbf{b}_i,
\end{aligned}$$

so $\mathbf{x}+\lambda\mathbf{y}\notin P$, thus $\mathbf{y}\notin\operatorname{rec}(P)$.

To show $\operatorname{lin}(P)=\{\mathbf{x}\in\mathbb{R}^d : A\mathbf{x}=0\}$, by definition of lineality space,

$$\operatorname{lin}(P)=\operatorname{rec}(P)\cap-\operatorname{rec}(P)=\{\mathbf{x}\in\mathbb{R}^d : A\mathbf{x}\le 0, A\mathbf{x}\ge 0\}=\{\mathbf{x}\in\mathbb{R}^d : A\mathbf{x}=0\}.$$

□

Exercise 3. Let $A \in \mathbb{R}^{m\times d}, \mathbf{b} \in \mathbb{R}^m$. Consider the polyhedron $P = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \le \mathbf{b}\}$. Show that P is bounded if and only if $\text{cone}(\{\mathbf{a}^1, \dots, \mathbf{a}^m\}) = \mathbb{R}^d$, where $\mathbf{a}^i$ is the ith row of A.

Proof. Notice that P is closed and convex since P is the intersection of halfspaces, then Theorem 2.4.21 in lecture notes yields that

$$P \text{ is bounded} \iff \text{rec}(P) = \{0\},$$

so we just need to show

$$\text{rec}(P) = \{0\} \iff \text{cone}(\{\mathbf{a}^1, \dots, \mathbf{a}^m\}) = \mathbb{R}^d.$$

Let $X = \{\mathbf{a}^1, \dots, \mathbf{a}^m\}$, by Exercise 6 in session 2, we have

$$(X^\bullet)^\bullet = \text{cl}(\text{cone}(X)) = \text{cone}(X),$$

where the second equality follows from Proposition 2.2.15 in lecture notes, and one can easily derive that

$$(X^\bullet)^\bullet = \mathbb{R}^d \iff X^\bullet = \{0\}.$$

Therefore, we just need to show

$$\text{rec}(P) = \{0\} \iff X^\bullet = \{0\}.$$

Observe that $\text{rec}(P) = X^\bullet$ since Exercise 2 shows $\text{rec}(P) = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \le 0\} = \{\mathbf{x} \in \mathbb{R}^d : \langle \mathbf{a}^i, \mathbf{x}\rangle \le 0,\ i = 1, \dots, m\}$, this completes the proof. □

Exercise 4. Let $C \subseteq \mathbb{R}^d$ be a compact, convex set. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a linear function given by $f(\mathbf{y}) = \langle \mathbf{a}, \mathbf{y}\rangle$ for some $\mathbf{a} \in \mathbb{R}^d$. Show that there exists an extreme point $\mathbf{v} \in C$ such that $f(\mathbf{v}) \le f(\mathbf{x})$ for every $\mathbf{x} \in C$.

Proof. f is a continuous function by definition, then Weierstrass theorem yields that there is some point $\mathbf{u} \in C$ such that $f(\mathbf{u}) \le f(\mathbf{x})$ for all $\mathbf{x} \in C$. Since C is compact and convex, by Krein-Milman $C = \text{conv}\,(\text{ext}(C))$. Then for some $\mathbf{v}^i \in \text{ext}(C)$ and $\lambda_i \ge 0$ summing to 1, we have $\mathbf{u} = \sum_{i=1}^k \lambda_i \mathbf{v}^i$ and

$$\langle \mathbf{a}, \mathbf{u}\rangle = \sum_{i=1}^k \lambda_i \left\langle \mathbf{a}, \mathbf{v}^i \right\rangle \ge \sum_{i=1}^k \lambda_i \langle \mathbf{a}, \mathbf{u}\rangle = \langle \mathbf{a}, \mathbf{u}\rangle.$$

Thus the inequality must be satisfied with equality, and $\langle \mathbf{a}, \mathbf{v}^i\rangle = \langle \mathbf{a}, \mathbf{u}\rangle$ for each of these $\mathbf{v}^i$. □