553.665 Introduction to Convexity, Fall 2022
Section 5

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Recall Farkas' lemma:

Theorem 1 (Farkas' Lemma)**.** *Let $A \in \mathbb{R}^{d\times n}$ and $\mathbf{b} \in \mathbb{R}^d$. Exactly one of the following is true.*

1. $A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq 0$ *has a solution.*

2. $\exists \mathbf{u} \in \mathbb{R}^d$ *such that* $\mathbf{u}^T A \leq 0$ *and* $\mathbf{u}^T \mathbf{b} > 0$*.*

Exercise 1. Let $A \in \mathbb{R}^{d\times n}, \mathbf{b} \in \mathbb{R}^d$. Exactly one of the following is true.

1. $A\mathbf{x} \leq \mathbf{b}$ has a solution.
2. $\exists \mathbf{u} \geq 0$ such that $\mathbf{u}^T A = 0$ and $\mathbf{u}^T \mathbf{b} < 0$.

Proof. Let $\mathbf{x}_i^+ = \max\{0, \mathbf{x}_i\} \geq 0,\ \mathbf{x}_i^- = -\min\{0, \mathbf{x}_i\} \geq 0,\ i = 1, \ldots, n.$ Then $\mathbf{x} = \mathbf{x}^+ - \mathbf{x}^-$, and

$$\begin{aligned} A\mathbf{x} \leq \mathbf{b} \text{ has no solution } &\iff A\mathbf{x} + \mathbf{s} = \mathbf{b},\ \mathbf{s} \geq 0 \text{ has no solution} \\ &\iff A(\mathbf{x}^+ - \mathbf{x}^-) + \mathbf{s} = \mathbf{b},\ \mathbf{s}, \mathbf{x}^+, \mathbf{x}^- \geq 0 \text{ has no solution} \\ &\iff \begin{bmatrix} A & -A & \mathbf{I} \end{bmatrix} \mathbf{y} = \mathbf{b},\ \mathbf{y} \geq 0 \text{ has no solution} \\ &\iff \exists \tilde{\mathbf{u}} \in \mathbb{R}^d \text{ such that } \tilde{\mathbf{u}}^T \begin{bmatrix} A & -A & \mathbf{I} \end{bmatrix} \leq 0 \text{ and } \tilde{\mathbf{u}}^T \mathbf{b} > 0 \end{aligned}$$

that is, $\tilde{\mathbf{u}}^T A \leq 0,\ -\tilde{\mathbf{u}}^T A \leq 0,\ \tilde{\mathbf{u}}^T \mathbf{I} \leq 0,\ \tilde{\mathbf{u}}^T \mathbf{b} > 0$, which gives $\tilde{\mathbf{u}}^T A = 0,\ \tilde{\mathbf{u}} \leq 0,\ \tilde{\mathbf{u}}^T \mathbf{b} > 0$. Then $\mathbf{u} = -\tilde{\mathbf{u}}$ is the desired vector. ☐

Exercise 2. Let $A \in \mathbb{R}^{d\times n}$. Exactly one of the following is true.

1. $A\mathbf{x} = 0,\ \mathbf{x} \geq 0$ has a nontrivial solution.
2. $\mathbf{y}^T A > 0$ has a solution.

Proof. We just need to prove

$$\exists \mathbf{x} \neq 0, \mathbf{x} \geq 0 \text{ such that } A\mathbf{x} = 0 \iff \nexists \mathbf{y} \in \mathbb{R}^d \text{ satisfying } \mathbf{y}^T A > 0.$$

Consider any fixed vector $\mathbf{b} > 0 \in \mathbb{R}^n$, then

$$\begin{aligned} \nexists \mathbf{y} \in \mathbb{R}^d \text{ satisfying } \mathbf{y}^T A > 0 &\iff \nexists \mathbf{y} \in \mathbb{R}^d \text{ such that } \mathbf{y}^T A \geq \mathbf{b} \\ &\iff A^T(-\mathbf{y}) \leq -\mathbf{b} \text{ has no solution} \\ &\overset{Ex.1}{\iff} \exists \mathbf{x} \geq 0 \text{ such that } \mathbf{x}^T A^T = 0 \text{ and } \mathbf{x}^T(-\mathbf{b}) < 0 \\ &\iff \exists \mathbf{x} \geq 0 \text{ such that } A\mathbf{x} = 0 \text{ and } \mathbf{x}^T \mathbf{b} > 0 \\ &\iff \exists \mathbf{x} \neq 0,\ \mathbf{x} \geq 0 \text{ such that } A\mathbf{x} = 0. \end{aligned}$$

$\square$

Exercise 3. Let $C \subseteq \mathbb{R}^d$ be a nonempty closed convex set. Then C has at least one extreme point if and only if C is pointed.

Proof. Let $\mathbf{x}$ be an extreme point of C. Suppose C is not pointed, then for any $\mathbf{r} \in \text{lin}(C)\backslash\{0\}$, notice that $\mathbf{x}+\mathbf{r},\ \mathbf{x}-\mathbf{r} \in C$ and $\frac{1}{2}((\mathbf{x}+\mathbf{r})+(\mathbf{x}-\mathbf{r})) = \mathbf{x}$, contradicting that $\mathbf{x}$ is an extreme point.

Conversely, we prove by induction on the dimension of the space to show that if C does not contain a line, then it must have an extreme point. It is trivial for the case when $d = 1$, so assume it is true for $d-1$ with $d \geq 2$. Then for any nonempty closed convex set $C \subseteq \mathbb{R}^d$, there must exist points $\mathbf{x} \in C$ and $\mathbf{y} \notin C$ since $\text{lin}(C) = \{0\}$. The line segment connecting $\mathbf{x}$ and $\mathbf{y}$ intersects the relative boundary of C at some point $\bar{\mathbf{x}}$ (see Figure 1). Consider a supporting hyperplane H of C passing through $\bar{\mathbf{x}}$, then $C \cap H$ lies in a $(d-1)$-dimensional space and does not contain a line. Hence, by induction hypothesis, $C \cap H$ must have an extreme point, and this extreme point must also be an extreme point of C. $\square$

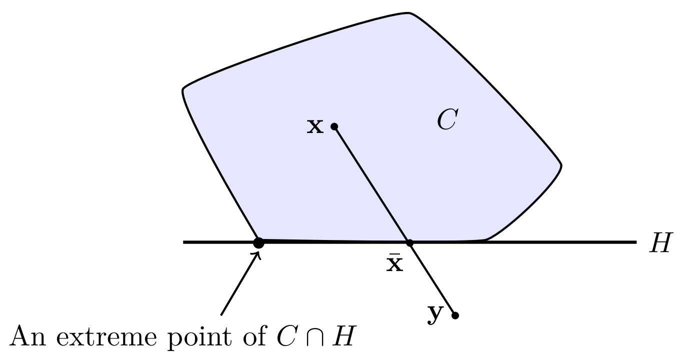

Figure 1: Illustration of the proof in Exercise 3

Corollary 1. *Every nonempty polyhedron* $P = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq 0\}$ *has at least one extreme point.*

Proof. We just need to show that P is pointed. Observe that

$$P = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} = \mathbf{b},\ \mathbf{x} \geq 0\} = \left\{\mathbf{x} \in \mathbb{R}^d : \begin{bmatrix} A \\ -A \\ -\mathbf{I} \end{bmatrix} \mathbf{x} \leq \begin{bmatrix} \mathbf{b} \\ -\mathbf{b} \\ \mathbf{0} \end{bmatrix}\right\},$$

then by Exercise 2 in Section 4,

$$\text{lin}(P) = \left\{\mathbf{x} \in \mathbb{R}^d : \begin{bmatrix} A \\ -A \\ -\mathbf{I} \end{bmatrix} \mathbf{x} = 0\right\} = \{0\}.$$

$\square$

Exercise 4. Let $A \in \mathbb{R}^{m \times d}, \mathbf{b} \in \mathbb{R}^m$. Consider the polyhedron $P = \{\mathbf{x} \in \mathbb{R}^d : A\mathbf{x} \leq \mathbf{b}\}$. Suppose there is some $\bar{\mathbf{x}} \in \mathbb{R}^d$ such that $A\bar{\mathbf{x}} < b$, show that $\dim(P) = d$.

Proof. For any $i \in \{1, \ldots, d\}$, there exists $\varepsilon_i > 0$ such that $A(\bar{\mathbf{x}} + \varepsilon_i \mathbf{e}^i) \leq \mathbf{b}$, where $\mathbf{e}^i$ is the ith standard unit vector. Then

$$\{\bar{\mathbf{x}}, \bar{\mathbf{x}} + \varepsilon_1 \mathbf{e}^1, \ldots, \bar{\mathbf{x}} + \varepsilon_d \mathbf{e}^d\}$$

is a set of $d+1$ affinely independent points in P, so $\dim(P) = d$. $\square$