553.665 Introduction to Convexity, Fall 2022
Section 2

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1 Review

Theorem 1 (Equivalent definitions of relative interior)**.** *Let $C \subseteq \mathbb{R}^d$ and $\mathbf{x} \in C$. The following are equivalent.*

1. $\mathbf{x} \in \mathit{relint}(C)$.
2. *There exists* $\varepsilon > 0$ *such that* $B(\mathbf{x}, \varepsilon) \cap \mathit{aff}(C) \subseteq C$.
3. *There exists* $\varepsilon > 0$ *such that* $\forall \mathbf{y} \in \mathit{aff}(C)\,, \mathbf{x} + \varepsilon \left(\frac{\mathbf{y}-\mathbf{x}}{\|\mathbf{y}-\mathbf{x}\|} \right) \in C$.
4. $\forall \mathbf{y} \in \mathit{aff}(C)\,,\ \exists \varepsilon_{\mathbf{y}} > 0$ *such that* $\mathbf{x} + \varepsilon_{\mathbf{y}}(\mathbf{y} - \mathbf{x}) \in C$.

Question 1. Let $X \subseteq \mathbb{R}^d$ be a compact convex set, is it true that cone(X) is closed?
Answer. No, a counterexample is provided in Figure 1.

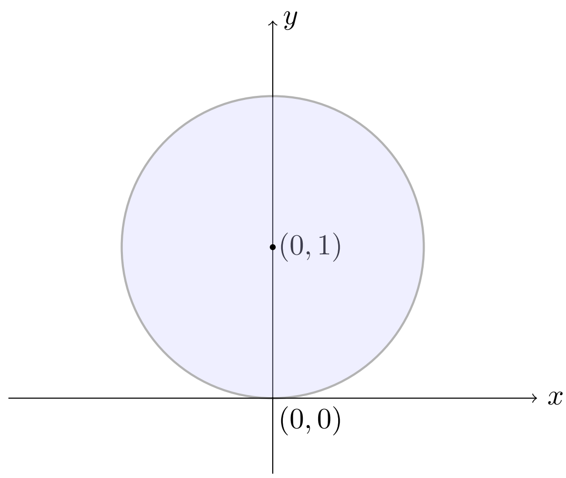

Figure 1: $X = \{(x, y) : x^2 + (y-1)^2 \leq 1\}$, but $\text{cone}(X) = \{(x, y) : y > 0\} \cup \{(0, 0)\}$ is not closed.

Question 2. Let $X, Y \subseteq \mathbb{R}^d$, is it true that $X \subseteq Y$ implies $\operatorname{relint}(X) \subseteq \operatorname{relint}(Y)$?
Answer. No, a counterexample is provided in Figure 2.

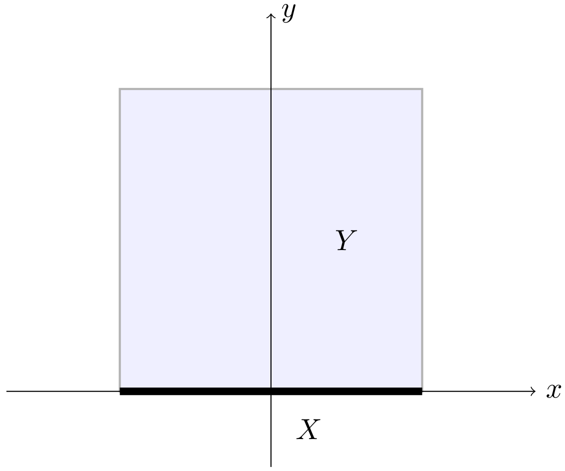

Figure 2: $X = \{(x, y) : x \in [-1, 1], y = 0\}$, $Y = \{(x, y) : x \in [-1, 1], y \in [0, 2]\}$, then $X \subseteq Y$ but $\operatorname{relint}(X) \cap \operatorname{relint}(Y) = \varnothing$.

2 Exercises

Exercise 1. Prove that the relative interior of a nonempty convex set is nonempty.

Proof. Hint: Let C be any nonempty convex set with dimension $d \geq 0$, then one can construct a point lie in $\operatorname{relint}(C)$ by some affinely independent points in C. Left as Homework. $\square$

Exercise 2. Let $X \subseteq \mathbb{R}^d$, and let $\mathbf{y} \in \operatorname{conv}(X)$. Suppose H is a halfspace such that $\mathbf{y} \in H$. Prove that $H \cap X \neq \varnothing$.

Proof. Let $H = \{\mathbf{x} : \langle \mathbf{a}, \mathbf{x} \rangle \leq \delta\}$ for some $\mathbf{a} \in \mathbb{R}^d$, $\delta > 0$. Since $\mathbf{y} \in \operatorname{conv}(X)$, we have $\mathbf{y} = \sum_{i=1}^k \lambda_i \mathbf{x}^i$ for some $\mathbf{x}^i \in X$, $\lambda_i \geq 0$, $i = 1, \ldots, k$ and $\sum_{i=1}^k \lambda_i = 1$. Now if $H \cap X = \varnothing$, then we would have $\langle \mathbf{a}, \mathbf{x}^i \rangle > \delta$, $\forall i \in \{1, \ldots, k\}$, but this would give $\langle \mathbf{a}, \mathbf{y} \rangle > \delta$, which leads to a contradiction. $\square$

Exercise 3. Let $X \subseteq \mathbb{R}^d$. Suppose H is a halfspace such that $X \subseteq H$. Prove that

$$\operatorname{conv}(H^= \cap X) = H^= \cap \operatorname{conv}(X),$$

where $H^=$ is the hyperplane associated with H.

Note: Talk about the relationship with MILPs, as well as the exercise 11 in session 1.

Proof. Let $H = \{\mathbf{x} : \langle \mathbf{a}, \mathbf{x} \rangle \leq \delta\}$ for some $\mathbf{a} \in \mathbb{R}^d$, $\delta > 0$, and $H^= = \{\mathbf{x} : \langle \mathbf{a}, \mathbf{x} \rangle = \delta\}$.

$\forall \mathbf{x} \in H^= \cap \operatorname{conv}(X)$, we have $\langle \mathbf{a}, \mathbf{x} \rangle = \delta$, $\mathbf{x} = \sum_{i=1}^k \lambda_i \mathbf{x}^i$ for some $\mathbf{x}^i \in X$, $\lambda_i \geq 0$, $i = 1, \ldots, k$ and $\sum_{i=1}^k \lambda_i = 1$. Therefore,

$$\sum_{i=1}^k \lambda_i (\langle \mathbf{a}, \mathbf{x}^i \rangle - \delta) = 0.$$

Observe that $\lambda_i (\langle \mathbf{a}, \mathbf{x}^i \rangle - \delta) \leq 0$, $\forall i \in \{1, \ldots, k\}$, then for each i, either $\lambda_i = 0$ or $\langle \mathbf{a}, \mathbf{x}^i \rangle = \delta$. Therefore, for those i such that $\lambda_i \neq 0$, $\mathbf{x}^i \in H^=$. Thus, $\mathbf{x} \in \operatorname{conv}(H^= \cap X)$.

To show the reverse inclusion, $\forall \mathbf{x} \in \operatorname{conv}(H^= \cap X)$, $\mathbf{x} = \sum_{i=1}^k \lambda_i \mathbf{x}^i$ for some $\mathbf{x}^i \in H^= \cap X,\ \lambda_i \geq 0,\ i = 1, \dots, k$ and $\sum_{i=1}^k \lambda_i = 1$. This implies $\mathbf{x} \in \operatorname{conv}(X)$ and $\mathbf{x} \in H^=$. That is, $\mathbf{x} \in H^= \cap \operatorname{conv}(X)$. ☐

Apply Exercise 3, we can provide an alternate proof of Theorem 2.

Theorem 2 (Carathéodory's theorem (convex version))**.** *Let $X \subseteq \mathbb{R}^d$ and $\mathbf{x} \in conv(X)$. Then $\mathbf{x}$ is a convex combination of at most $d+1$ points of X.*

Proof. Base case: It's easy to check when $d = 1$.
Induction step: Suppose it's true for all dimensions less than d. $\mathbf{x} \in \operatorname{conv}(X)$, then by definition there exist distinct $\mathbf{x}^1, \dots, \mathbf{x}^k$ such that $\mathbf{x} \in \operatorname{conv}\left(\{\mathbf{x}^1, \dots, \mathbf{x}^k\}\right) := C$.

Case 1: $\mathbf{x} \in \operatorname{relbd}(C)$, then let $H^=$ be a supporting hyperplane of C through $\mathbf{x}$, then

$$\mathbf{x} \in H^= \cap C = H^= \cap \operatorname{conv}\left(\{\mathbf{x}^1, \dots, \mathbf{x}^k\}\right) \overset{\textit{Exercise 3}}{=} \operatorname{conv}\left(H^= \cap \{\mathbf{x}^1, \dots, \mathbf{x}^k\}\right).$$

Since $H^=$ is $(d-1)$-dimensional, by induction hypothesis, $\mathbf{x}$ is a convex combination of at most d points of $H^= \cap \{\mathbf{x}^1, \dots, \mathbf{x}^k\}$, in particular of at most d points of X.

Case 2: $\mathbf{x} \in \operatorname{relint}(C)$, observe that there exists some $i \in \{1, \dots, k\}$ such that $\mathbf{x} \neq \mathbf{x}^i$, then let $\mathbf{y} \in \{\mathbf{x}^i + \lambda(\mathbf{x} - \mathbf{x}^i) : \lambda > 0\} \cap \operatorname{relbd}(C)$. Therefore, $\mathbf{x}$ is a convex combination of at most $d+1$ points of X since $\mathbf{x}$ is a convex combination of $\mathbf{y}$ and $\mathbf{x}^i$, and $\mathbf{y}$ is a convex combination of at most d points of X. ☐

Definition 1. Let $X \subseteq \mathbb{R}^d$ be a linear subspace. We define $X^\perp := \{\mathbf{y} \in \mathbb{R}^d : \langle \mathbf{y}, \mathbf{x} \rangle = 0,\ \forall \mathbf{x} \in X\}$ as the *orthogonal complement* of X.

Definition 2. Let $X \subseteq \mathbb{R}^d$ be any set. We define

$$X^\circ := \{\mathbf{y} \in \mathbb{R}^d : \langle \mathbf{y}, \mathbf{x} \rangle \leq 1,\ \forall \mathbf{x} \in X\},$$

$$X^\bullet := \{\mathbf{y} \in \mathbb{R}^d : \langle \mathbf{y}, \mathbf{x} \rangle \leq 0,\ \forall \mathbf{x} \in X\}.$$

Exercise 4. Let $X \subseteq \mathbb{R}^d$ be a convex cone, prove that $X^\circ = X^\bullet$.

Proof. It's easy to check that $X^\bullet \subseteq X^\circ$.

To show the reverse inclusion, $\forall \mathbf{y} \in X^\circ$, if $\langle \mathbf{y}, \mathbf{x} \rangle > 0$ for some $\mathbf{x} \in X$, then

$$\langle \mathbf{y}, \underbrace{\frac{2}{\langle \mathbf{y}, \mathbf{x} \rangle} \mathbf{x}}_{\in X} \rangle = 2 > 1$$

contradicts with $\mathbf{y} \in X^\circ$. ☐

Remark 1. Similarly, one can prove that $\operatorname{cone}(X)^\circ = X^\bullet$ for any $X \subseteq \mathbb{R}^d$.

Exercise 5. Let $X \subseteq \mathbb{R}^d$ be a linear subspace, prove that $X^\circ = X^\perp$.

Proof sketch. Similar to the proof of Exercise 4. $\forall \mathbf{y} \in X^\circ$, if $\langle \mathbf{y}, \mathbf{x} \rangle \neq 0$ for some $\mathbf{x} \in X$, then $\langle \mathbf{y}, \frac{2}{\langle \mathbf{y}, \mathbf{x} \rangle} \mathbf{x} \rangle = 2 > 1$ contradicts with $\mathbf{y} \in X^\circ$. ☐

Exercise 6. Let $X \subseteq \mathbb{R}^d$ be any set, prove that $(X^\bullet)^\bullet = \text{cl}(\text{cone}(X))$.

Proof. First we show that $X^\bullet$ is a convex cone. $\forall \mathbf{a}, \mathbf{b} \in X^\bullet, \lambda, \gamma \geq 0 \implies \forall \mathbf{x} \in X, \langle \lambda\mathbf{a} + \gamma\mathbf{b}, \mathbf{x}\rangle = \lambda\langle \mathbf{a}, \mathbf{x}\rangle + \gamma\langle \mathbf{b}, \mathbf{x}\rangle \leq 0 \implies \lambda\mathbf{a} + \gamma\mathbf{b} \in X^\bullet \implies X^\bullet$ is a convex cone. Therefore,

$$\begin{aligned}
(X^\bullet)^\bullet &= (X^\bullet)^\circ && (1)\\
&= (\text{cone}(X)^\circ)^\circ && (2)\\
&= \text{cone}(X)^{\circ\circ}\\
&= \text{cl}(\text{conv}(\text{cone}(X) \cup \{0\})) && (3)\\
&= \text{cl}(\text{cone}(X)),
\end{aligned}$$

where (1) follows from Exercise 4, (2) follows from Remark 1, (3) follows from Proposition 2.4.9 (2) in lecture notes. □