553.385 Numerical Linear Algebra, Spring 2022
Section 4

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1 Analytic properties of vector norms

Definition 1.1. *Let $V = \mathbb{R}^n$ or $\mathbb{C}^n$ with a given norm $\|\cdot\|$, we say that a sequence $\{x_k\}$ of vectors in V converges to a vector $x \in V$ with respect to $\|\cdot\|$ if and only if $\lim_{k\to\infty} \|x_k - x\| = 0$. If $\{x_k\}$ converges to x with respect to $\|\cdot\|$, we write $\lim_{k\to\infty} x_k = x$ with respect to $\|\cdot\|$.*

Exercise 1: Can a sequence of vectors converge to two different limits with respect to a given norm?

If $\lim_{k\to\infty} x_k = x$ and $\lim_{k\to\infty} x_k = y$, then prove $x = y$.

Theorem 1.1. *If $\|\cdot\|_\alpha$ and $\|\cdot\|_\beta$ are norms on a finite-dimensional real or complex vector space V, and if $\{x_k\}$ is a given sequence of vectors in V, then $\lim_{k\to\infty} x_k = x$ with respect to $\|\cdot\|_\alpha$ if and only if $\lim_{k\to\infty} x_k = x$ with respect to $\|\cdot\|_\beta$.*

Proof. Since all norms in $\mathbb{R}^n$ or $\mathbb{C}^n$ are equivalent, we have

$$c_1 \|x_k - x\|_\alpha \leq \|x_k - x\|_\beta \leq c_2 \|x_k - x\|_\alpha, \ \forall k \in \mathbb{N}_+$$

It follows that $\|x_k - x\|_\alpha \to 0$ if and only if $\|x_k - x\|_\beta \to 0$ as $k \to \infty$. □

Exercise 2: Prove $\lim\limits_{p\to\infty} \|x\|_p = \|x\|_\infty$.

Definition 1.2 (Cauchy sequence)**.** *A sequence $\{x_k\}$ in a vector space V is a Cauchy sequence with respect to a norm $\|\cdot\|$ if for each $\varepsilon > 0$ there is a positive integer $N(\varepsilon)$ such that $\|x_{k_1} - x_{k_2}\| \leq \varepsilon$ when $k_1, k_2 \geq N(\varepsilon)$.*

Theorem 1.2. *Let $\|\cdot\|$ be a given norm on $\mathbb{R}^n$ or $\mathbb{C}^n$, and let $\{x_k\}$ be a given sequence of vectors in V. The sequence $\{x_k\}$ converges to a vector in V if and only if it is a Cauchy sequence with respect to the norm $\|\cdot\|$*

Definition 1.3. *A normed linear space V is said to be complete with respect to its norm $\|\cdot\|$ if every sequence in V that is a Cauchy sequence with respect to $\|\cdot\|$ converges to a point of V.*

2 Matrices (review)

Definition 2.1 (Induced matrix norm)**.** *The norm $\|A\| = \sup_{\|x\|<1} \|A\mathbf{x}\|$ is called the induced matrix norm associated to the vector norm $\|\cdot\|$.*

Definition 2.2 (Consistency)**.** *A matrix norm $\|\cdot\|$ on $\mathbb{E}(n,m)$ is said to be consistent with vector norms $\|\cdot\|_a$ and $\|\|_b$ on $\mathbb{C}^n$ if*

$$\|A\mathbf{x}\|_a \leq \|A\| \cdot \|\mathbf{x}\|_b$$

Definition 2.3 (Proper)**.** *A matrix norm $\|\cdot\|$ on $\mathbb{E}(n,n)$ is said to be proper if*

$$\|AB\| \leq \|A\| \cdot \|B\|$$

Exercise 3: Prove that the Frobenius norm is a proper matrix norm and that it is consistent with the ℓ_2 vector norm $\|\cdot\|_2$.

Theorem 2.1 (Convergence of Geometric Series)**.** *Let $A \in \mathbb{E}(n,n)$, if $\rho(A) < 1$ then $(I-A)^{-1}$ exists and*

$$(I-A)^{-1} = I + A + A^2 + A^3 + \cdots$$

Exercise 4: What will happen if A is a strictly lower triangular matrix (all the entries on the main diagonal of a lower triangular matrix are also 0)?